AN ELEMENTARY EXPOSITION

OF

Grassmann's Ausdehnungslehre,

OR,

THEORY OF EXTENSION

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AN ELEMENTARY EXPOSITION OF GRASSMANN'S "AUSDEHNUNGSLEHRE," OR THEORY OF EXTENSION.

Grassmann's *Ausdehnungslehre* is one of the few great works of mathematics of the nineteenth century. Appearing first in 1844 and rewritten in 1862, it is only within the last decade or two that it has received a tardy recognition. One reason for this is found in the difficulty of the subject itself, being unlike other mathematics; and another, in the rigorous methods of presentation adopted by the author. In the *Ausdehnungslehre* of 1862, following some 150 pages of theory, the author for the first time gives his subject concrete form by applying his method to geometry. The theoretical part is naturally the more difficult, while the application to geometry is the more interesting. Hyde, in his *Directional Calculus*, purposing to present the *Ausdehnungslehre* to American readers, cut the knot of the difficulty by taking the results of the theoretical part for granted and giving only the application to geometry, and by limiting his treatment to two and three dimensional space.

An elementary exposition which will give the simpler portions of the theoretical part as well as the applications of the theory seems to be needed. Such an exposition should serve the needs of two classes of readers: first, of those who would like to have a good general idea of the subject without going very deeply into its particulars; and secondly, of those who, expecting to make a thorough study of the subject, wish first to read an introduction to it. To meet this want the following pages have been written. In some places for the purpose of making the subject clearer, changes have been made, but wherever they have been introduced, attention is always called to them.

CHAPTER I.

INTRODUCTION.

1. In elementary mathematics only one kind of unit is admitted, or at most two, viz., 1 and $\sqrt{-1}$. In the Theory of Extension besides the absolute unit of arithmetic and algebra, *extensive quantities* appear. The extensive quantities are different in nature from the absolute unit and different from each other. As a simple example of an extensive quantity we may name a *vector* (§3).

2. A *Scalar* quantity is a quantity of elementary mathematics, *i. e.*, a simple number, either positive or negative.

3. A *Vector* is a straight line whose length and direction are fixed but not its position. Thus any two parallel and equal straight lines may represent the same vector. A vector gives the relative position of one point with reference to another, viz., a certain distance in a certain direction.

4. Vectors can be *added* and *subtracted*.

Thus if ε_1 is the vector from O to A and ε_2 the vector from A to B, the sum of ε_1 and ε_2 is ε_3, because translation from O to B along the straight line OB is equivalent or equal in the vector sense, to translation along OA and AB. Thus,

$$\varepsilon_1+\varepsilon_2=\varepsilon_3.$$

Transposing, we get

$$\varepsilon_1=\varepsilon_3-\varepsilon_2=\varepsilon_3+(-\varepsilon_2).$$

Interpreting this equation we see that translation along ε_3 followed by translation along ε_2 *in the negative direction* is equal to translation along ε_1.

The sum of any number of vectors may evidently be found in the same way. Thus, in the figure

$$\varepsilon_5=\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4.$$

Stated generally, we have

The sum of any number of vectors is found by joining the beginning point of the second vector to the end point of the first, the beginning point of the third to the end point of the second, and so on; the vector from the beginning point of the first vector to the end point of the last is the sum required.

The sum and difference of two vectors are the diagonals of the parallelogram whose adjacent sides are the given vectors.

Or, more explicitly—

(1). *The sum of two vectors going out from an origin and forming two sides of a parallelogram is that dianonal of the parallelogram which passes through the origin.*

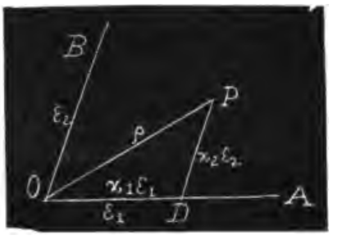
(2). *The difference of two such vectors is that dianonal which proceeds from the end of the subtrahend vector to the end of the minuend vector.*

5. Vectors and line segments give us simple examples of extensive quantities. We proceed to show how lines and vectors can be used in a system of coördinates.

6. The simplest case of this is where a point is located on a given line. Let ρ denote any given line, and let O be an origin on it. Let further x be a scalar. Then by giving the proper value to x, $x\rho$ will locate any point P whatever on the line. Here ρ is to be regarded as an extensive quantity since it denotes not a number but the position and length of a line.

7. The next simplest case of coördinates is that in which a point is located in a plane by means of two vectors.

Let O, the origin, be a point in the given plane, and ϵ_1 and ϵ_2 two unit vectors in this plane.



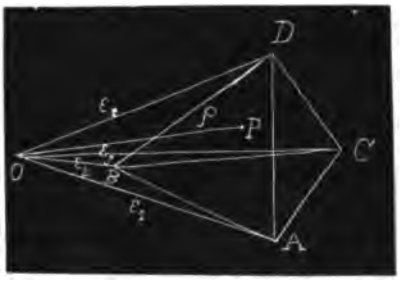
Then, by making

$$\rho = x_1\epsilon_1 + x_2\epsilon_2$$

where $x_1 = \frac{r\sin BOP}{\sin BOA}$, $x_2 = \frac{r\sin POA}{\sin BOA}$,

and r=length of ρ, the point P may be located at any point of the plane. Here ϵ_1, ϵ_2 and ρ are extensive quantities, and x_1 and x_2 are scalars.

8. In space we may have a similar system containing three vectors. Thus, if



$$\rho = x_1\epsilon_1 + x_2\epsilon_2 + x_3\epsilon_3$$

by assigning values to x_1, x_2, and x_3, P as the extremity of ρ may be located at any point in space.

9. In the same way we can have a system including four vectors. Thus, if P is any point, ϵ_1, ϵ_2, ϵ_3, ϵ_4 are four unit vectors, and

$$x_1 = \frac{\text{pyramid}P-BDC}{\text{pyramid}A-BDC}, \quad x_2 = \frac{\text{pyramid}P-ADC}{\text{pyramid}B-ADC},$$

etc., we have the equation

$$\rho = x_1\epsilon_1 + x_2\epsilon_2 + x_3\epsilon_3 + x_4\epsilon_4,$$

i. e. translation along ϵ_1 a distance equal to x_1, followed by translation along ϵ_2 a distance equal to x_2, and so on, is equivalent to translation from O to P direct. A geometrical proof of the truth of this can be given, but it is not thought necessary to insert it here.

REMARK. In the preceding the ϵ's in a certain sense denote dimensions. Then the *space* considered in this last article is of the *fourth* order.

CHAPTER II.

ADDITION, SUBTRACTION, MULTIPLICATION, AND DIVISION OF EXTENSIVE QUANTITIES.

10. DEFINITION. A quantity is said to be *independent* when it can not be expressed in terms of others. A quantity is said to be *dependent* when it can be numerically expressed in terms of others, *i. e.* as a sum formed out of numerical multiples of these quantities.

Thus if a_1, a_2, ... are extensive quantities, and α_1, α_2, ... are

real numbers, positive or negative, and

$$a=\alpha_1 a_1+\alpha_2 a_2+\alpha_3 a_3+\ldots\ldots$$

a is said to be a dependent quantity and to be '*numerically expressed*' in terms of $a_1, a_2, a_3, \ldots\ldots$

11. A quantity a_1 is a *Unit* if it can serve along with other like units, a_2, a_3, ... to give a series of numerically derived quantities, a...... A unit is said to be *original* if it is not derived from other units. A set of quantities which are independent, *i. e.*, no one of which is numerically expressible in terms of one or more of the others is called a *System of Units*, provided any number of other quantities can be expressed in terms of them.

As an illustration of such a system we may take the set of units given in Art. 8. There $\epsilon_1, \epsilon_2, \epsilon_3$ are a set of quantities which are independent because any sum formed out of multiples of ϵ_1 and ϵ_2 can never be a quantity like ϵ_3, since any sum formed from ϵ_1 and ϵ_2 would be a quantity in the plane of these two (7) while ϵ_3 is outside of this plane. Moreover, any number of other quantities, ρ's, can be derived from $\epsilon_1, \epsilon_2, \epsilon_3$.

12. Definition. An Extensive *Quantity* is a quantity numerically derived from a *system* of units. If an extensive quantity can be derived from the original units it is called an extensive quantity of the *first kind*.

13. Definition. Quantities from the same system can be added (or subtracted) by adding the numerical coefficients of the same units. Thus

$$\Sigma\alpha e+\Sigma\beta e=\Sigma(\alpha+\beta)e$$

where the α's and β's under the summation signs are numbers, and the e's are extensive units. We may add here that in the *Ausdehnungslehre* the distributive law is always assumed to hold.

Remark. To remove ambiguity it will be understood that all indicated operations are performed as one comes to them from the left. Thus $a+b+c$ means $(a+b)+c$, and abc means $(ab)c$.

14. The following formulas underlie and justify all the operations involved in addition and subtraction in algebra. They follow directly from the definition in 13.

(1). $a+b=b+a$, a commutative law in addition and subtraction.

(2). $a+(b+c)=a+b+c$, associative law in addition and subtraction.

$$\left.\begin{array}{l}(3).\ a+b-b=a\\(4).\ a-b+b=a\end{array}\right\}\textit{ opposite character of addition and subtraction.}$$

Hence all the laws for addition and subtraction of algebraic numbers hold also for extensive quantities.

15. Definition. When an extensive quantity is multiplied (or divided) by a *number* each of its coefficients is multiplied by that number.

Thus

$$\Sigma\alpha e.\beta=\Sigma(\alpha\beta)e.$$

REMARK. If a is an extensive quantity and α a number, then in αa or $a\alpha$ the numerical factor is the multiplier and the other factor is the multiplicand.

16. From Art. 15 we infer the following formulas :

$$
\begin{aligned}
&(1).\ a\alpha=\alpha a,\\
&(2).\ a\beta\gamma=a(\beta\gamma),\\
&(3).\ (a+b)\gamma=a\gamma+b\gamma,\\
&(4).\ a(\beta+\gamma)=a\beta+a\gamma,
\end{aligned}
$$

where, as heretofore, the Greek letters denote real numbers and the Roman letters, extensive quantities. From these formulas it follows—

That all the laws of multiplication and division of algebraic quantities hold also for extensive quantities multiplied or divided by numbers.

17. DEFINITION. The totality of quantities which are derivable from a series of extensive quantities, a_1, a_2, a_3, ...a_n is called the *Space* of those quantities. A space which can be formed out of not less than n such quantities each of the first kind (12) is called a space of the nth order.

18. DEFINITION. If every quantity of a space (A) is at the same time a quantity of another space (B), while the converse is not true, then the spaces are said to be *incident :* the first is said to be *subordinate* to the second, and the second, to *include* the first.

19. *If n independent quantities a_1......a_n can be numerically expressed in terms of n other quantities b_1......b_n, then is the space of the first quantities identical with that of the last quantities. But if the n quantities a_1......a_n can be expressed in terms of less than n quantities b_1......b_n, then a_1... ..a_n are not independent, and some of them can be numerically expressed in terms of others.*

20. *Two quantities of a space of the nth order are equal to each other when and only when their numerical coefficients of the same units are equal.* This is analogous to the algebraic theorem which says that two complex numbers are equal only when their real parts are equal and also their imaginary parts.

21. *If the coefficients x_1....x_n by which an extensive quantity x is expressed in terms of the units e_1....e_n satisfy an equation of the mth degree $f(x_1....x_n)=0$, then the coefficients y_1....y_n by which x is expressed in terms of a_1. ..a_n of the same space also satisfy an equation of the mth degree, and if the first equation is homogeneous, the latter is also.*

PROOF. Let $a_1=\Sigma\alpha_{1r}e_r$, ... Then we have

$$x_1e_1+x_2e_2+\ldots=y_1\Sigma\alpha_{1r}e_r+y_2\Sigma\alpha_{2r}e_r+\ldots=\Sigma y_r\alpha_{r1}.e_1+\Sigma y_r\alpha_{r2}.e_2+\ldots$$

$$\therefore\ x_1=\Sigma y_r\alpha_{r1},\ x_2=\Sigma y_r\alpha_{r2},\ \ldots\ \text{(Art. 20)}.$$

But if these values are substituted in $f(x_1, ...x_n)=0$, we get an equation of the mth degree in y_1, y_2, ..., and, indeed, homogeneous if the first equation is homogeneous.

CHAPTER III.

MULTIPLICATION OF EXTENSIVE QUANTITIES. DIFFERENT KINDS OF MULTIPLICATION.

22. In the multiplication of extensive quantities expressed in terms of *units*, it is assumed that the distributive law holds, and that numerical coefficients may be treated as in elementary algebra (16).

Thus if $a=\Sigma\alpha_r e_r$ and $b=\Sigma\beta_s e_s$ are two extensive quantities in which α and β are numbers and the e's are extensive *units*, we may write

$$ab=[\Sigma\alpha_r e_r . \Sigma\beta_s e_s]=\Sigma\alpha_r\beta_s[e_r e_s],$$

that is to say, in the result each term of the multiplier is multiplied into every term of the multiplicand, and the partial products are added.

Notice that the law is assumed to hold only when the two factors are sums of *units*. In the theorems which follow it is shown that the same law applies when the factors are sums of *quantities*.

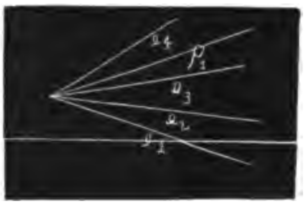
23. Before going on to the proofs of these theorems, we will illustrate the question involved by an example. (See Art. 9).

Let

$$\rho_1=r_{11}e_1+r_{12}e_2+r_{13}e_3+\ldots\ldots$$

$$\rho_2=r_{21}e_1+r_{22}e_2+r_{23}e_3+\ldots\ldots$$

Also, let



$$a=\alpha_1\rho_1+\alpha_2\rho_2+\ldots\ldots$$

$$b=\beta_1\rho_1+\beta_2\rho_2+\ldots\ldots$$

It is to be shown, then, that if $\rho_1\rho_2=\Sigma r_{1r}r_{2s}[e_r e_s]$, $ab=\Sigma\alpha_r\beta_s[\rho_r\rho_s]$.

The proof will be based on the definitions laid down and the theorems proved in the last chapter.

REMARK. It should be kept in mind in Articles 24—30 that α, β, ... denote numbers, the e's denote extensive *units* (11), and a, b, ... denote extensive *quantities* (12). Square brackets are used to indicate that the quantities inside are extensive quantities whose product is required.

24. *To show that* $[\Sigma\alpha_r e_r . b]=\Sigma\alpha_r[e_r b]$, *i. e.*, to show that in multiplying $\Sigma\alpha_r e_r$ by b each term of $\Sigma\alpha_r e_r$ is multiplied by b.

PROOF. Let $b=\Sigma\beta_s e_s$. Then

$$[\Sigma\alpha_r e_r . b]\equiv[\Sigma\alpha_r e_r . \Sigma\beta_s e_s]=\Sigma\alpha_r\beta_s[e_r e_s] \quad (22)$$

$$=\Sigma\alpha_1\beta_s[e_1 e_s]+\Sigma\alpha_2\beta_s[e_2 e_s]+\ldots\ldots(14)$$

$$=\alpha_1\Sigma\beta_s[e_1 e_s]+\alpha_2\Sigma\beta_s[e_2 e_s]+\ldots\ldots(15)$$

$$=\alpha_1[e_1.\Sigma\beta_s e_s]+\alpha_2[e_2.\Sigma\beta_s e_s]+ \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(22)$$

$$=\alpha_1[e_1.b]+\alpha_2[e_2.b]+\ldots.+\Sigma\alpha_r[e_r b].$$

25. *To show that* $[(a+b+\ldots.)p]=[ap]+[bp]+\ldots.$

$$[p(a+b+\ldots.)]=[pa]+[pb]+\ldots.$$

Proof. Let $a=\Sigma\alpha_r e_r$, $b=\Sigma\beta_r e_r\ldots.$ Then

$$[(a+b+\ldots.)p]=[(\Sigma\alpha_r e_r+\Sigma\beta_r e_r+\ldots.)p]=[\Sigma(\alpha_r+\beta_r+\ldots.)e_r.p]\ldots.(14)$$

$$=\Sigma(\alpha_r+\beta_r+\ldots.)[e_r p] \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(24)$$

$$=\Sigma[\alpha_r e_r\, p]+\Sigma[\beta_r e_r.p]+ \quad\ldots\ldots\ldots\ldots\ldots\ldots(14,\ 24)$$

$$=[ap]+[bp]+\ldots.$$

26. *To show that* $[(\alpha a)b]=\alpha[ab]$, and $[b(\alpha a)]=\alpha[ba]$.

Proof. Let $a=\Sigma\alpha_r e_r$. Then

$$[(\alpha a)b]\equiv[(\alpha\Sigma\alpha_r e_r)b]=[\Sigma\alpha\alpha_r e_r)b] \quad\ldots\ldots\ldots\ldots\ldots\ldots(15)$$

$$=\Sigma\alpha\alpha_r[e_r b]\ (24)\ =\alpha[\Sigma\alpha_r e_r.b]\ (16,\ 24)\ =\alpha[ab].$$

The other formula is obtained by making b the first factor in the above proof.

27. *To show that* $[(\alpha a+\beta b+\ldots.)p]=\alpha[ap]+\beta[bp]+\ldots.$

$$\text{and } [p(\alpha a+\beta b+\ldots.)]=\alpha[pa]+\beta[pb]+\ldots.$$

Proof. $[(\alpha a+\beta b+\ldots.)p]=[(\alpha a)p]+[(\beta b)p]+ \quad\ldots\ldots\ldots\ldots\ldots\ldots(25)$

$$=\alpha[ap]+\beta[bp]+ \quad\ldots\ldots\ldots\ldots\ldots\ldots(26).$$

28. We are now in position to show that the distributive law holds when the two factors are sums of multiples of extensive *quantities* as well as when they are sums of multiples of units.

Proof. $[\Sigma\alpha_r a_r.\Sigma\beta_s b_s]=\Sigma\alpha_r[a_r.\Sigma\beta_s b_s] \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(27)$

$$=\Sigma\alpha_r(\Sigma b_s[a_r b_s])\ (27)\ =\Sigma\alpha_r\beta_s[a_r b_s]\ldots\ldots\ldots\ldots(16).$$

This theorem holds also for any number of factors, as can be shown by mathematical induction.

29. Let us denote a product containing a number of factors, a, b, $\ldots.$

by $P_{a,b}$..... In such a product suppose a factor p equals $qa+rb+\ldots\ldots$ where $q, r, \ldots\ldots$ are numbers ; to show that

$$P_{qa+rb+\ldots\ldots}=q.P_a+r.P_{b+\ldots\ldots}$$

PROOF. However the product may be made up, we can always regard p as combined with another factor, then this product with other factors in turn. In each of the multiplications in which p enters as one factor Art. 27 applies.

30. *To show that* $P_{qa,\,rb,\,sc,}\ \ldots\ldots=qrs\ldots\ldots P_{a,\,b,\,c}\ \ldots\ldots$

PROOF. By 29, $P_{qa}=q\,P_a$. Then

$$P_{qa,\,rb,\,sc}\ \ldots\ldots=q.P_{a,\,rb,\,sc}\ \ldots\ldots=qrs\ldots\ldots P_{a,\,b,\,c}\ \ldots\ldots$$

It evidently follows that $P_{qa,\,ra}=P_{ra,\,qa}$.

31. *Different Kinds of Multiplication.*—Different kinds of multiplication are obtained by laying down different laws for simplifying a distributed product.

To illustrate :—

(1) $(\alpha_1e_1+\alpha_2e_2)(\beta_1e_1+\beta_2e_2)=\alpha_1\beta_1e_1^2+\alpha_1\beta_2e_1e_2+\alpha_2\beta_1e_2e_1+\alpha_2\beta_2e_2^2$.

In ordinary algebra the result is simplified by supposing $e_1e_2=e_2e_1$. The result in this way becomes

$$\alpha_1\beta_1e_1^2+(\alpha_1\beta_2+\alpha_2\beta_1)e_1e_2+\alpha_2\beta_2e_2^2,$$

or, say, $m_1e_1^2+m_2e_1e_2+m_3e_2^2$, where the m's are numerical coefficients.

(2) Similarly, writing three factors, we get,

$$(\alpha_1e_1+\alpha_2e_2)(\beta_1e_1+\beta_2e_2)(\gamma_1e_1+\gamma_2e_2)=m_1e_1^3+m_2e_1^2e_2+m_3e_1e_2^2+m_4e_2^3$$

by supposing, as in ordinary algebra, that

$$\begin{cases} e_1e_1e_2=e_1e_2e_1=e_2e_1e_1 \\ e_1e_2e_2=e_2e_1e_2=e_2e_2e_1 \end{cases}$$

Here the assumed law of simplification reduces a product which would otherwise have eight terms to one of four.

(3) The law of simplification in quaternions (which is a branch of mathematics using a certain kind of extensive quantities) may be seen by multiplying together two factors of three terms each.

Thus, $(\alpha_1e_1+\alpha_2e_2+\alpha_3e_3)(\beta_1e_1+\beta_2e_2+\beta_3e_3)$

$$=-(\alpha_1\beta_1+\alpha_2\beta_2+\alpha_3\beta_3)+(\alpha_1\beta_2-\alpha_2\beta_1)e_3+(\alpha_2\beta_3-\alpha_3\beta_2)e_1+(\alpha_3\beta_1-\alpha_1\beta_3)e_2,$$

by supposing $e_1^2=e_2^2=e_3^2=-1$, $e_1e_2=e_3$, $e_2e_3=e_1$, and $e_3e_1=e_2$.

32. Definition. A multiplication is said to be *linear* when the same laws of simplification of the distributed product continue to hold when numerically derived quantities (10) replace the given units.

33. *To show that there are but four kinds of linear multiplication.*

Let

$$(a) \qquad \Sigma\alpha_{rs}[e_r e_s]=0$$

express a simplification law in the product $\Sigma\alpha_r\beta_s[e_r e_s]$.

Let, now, e_r be replaced by $\Sigma x_{r,u}e_u$ and e_s by $\Sigma x_{sv}e_v$. We thus get

$$\Sigma\alpha_{rs}[\Sigma x_{ru}e_u\ \Sigma x_{sv}e_v]=0,$$

$$\text{whence, } \Sigma\alpha_{rs}\Sigma x_{ru}x_{sv}[e_u e_v]=0,\ (28),$$

$$\text{or, } \Sigma\alpha_{rs}x_{ru}x_{sv}[e_u e_v]=0.\ (16).$$

This equation is symmetrical in r and s and u and v and evidently will continue to hold true when these letters are interchanged. This gives

$$\Sigma\alpha_{sr}x_{sv}x_{ru}[e_v e_u]=0.$$

Adding the last two equations, we have

$$(b) \qquad \Sigma x_{ru}x_{sv}\{\alpha_{rs}[e_u e_v]+\alpha_{sr}[e_v e_u]\}=0.$$

Equation (b) may also be gotten by multiplying $\Sigma\alpha_{rs}x_{ru}x_{sv}[e_u e_v]$ by 2 and arranging the result with reference to equal coefficients, $(x_{rv}x_{sv})$. This derivation shows (b) to be a necessary, and not, as might appear, an arbitrary inference from the given equation.

Now from the nature of the case the coefficients x_{ru}, x_{sv} must be capable of having any values, as would the x's in Articles 6—9. If we assume that the *products* $x_{ru}x_{sv}$ are arbitrary, then from the theory of equations we have

$$(c) \qquad \alpha_{rs}[e_u e_v]+\alpha_{sr}[e_v e_u]=0,$$

true for all values of r and s and u and v.*

If we put $u=v$ in (c) we get

$$(d) \qquad (\alpha_{rs}+\alpha_{sr})[e_u e_u]=0.$$

This equation is satisfied either by assuming (1) $\alpha_{rs}+\alpha_{sr}=0$, *i. e.*, $\alpha_{rs}=-\alpha_{sr}$; or, by assuming (2), $[e_u e_v]=0$.

*Grassmann's derivation of (c) does not assume that the products $x_{ru}x_{sv}$ are arbitrary. The writer gave another demonstration which does not assume this before the Mathematical Section of the American Association for the Advancement of Science, 1899 meeting. Though somewhat simpler than Grassmann's proof, it would add materially to the length of this article.

(1) If $\alpha_{rs}=-\alpha_{sr}$, and we make this substitution in (c), there results

$$\alpha_{rs}\{[e_ue_v]-[e_ve_u]\}=0.$$

In this equation either $\alpha_{rs}=0$, or $[e_ue_v]=e_ve_u]$. If $\alpha_{rs}=0$, all the coefficients reduce to zero) and equation (a) vanishes identically, which is contrary to hypothesis. If

$$(e) \qquad [e_ue_v]-[e_ve_u]=0, \text{ or } [e_ue_v]=[e_ve_u],$$

we have the law for a form of multiplication of extensive quantities which is analogous to ordinary multiplication in algebra. See Art. 31, (1).

(2) If we say $[e_ue_u]=0$, it is equivalent to making in equation (a). $\alpha_{rr}=1$, and all the other coefficients equal to 0. Making this substitution in (c), we get

$$(f) \qquad [e_ue_v]+[e_ve_u]=0,$$

which implies $[e_ue_u]=0$, as may be seen by making $u=v$ in (f).

We have seen that equations (e) and (f) are necessary conditions in order that a multiplication may be linear. That they are sufficient conditions may be seen as follows: If we start with

$$(a) \qquad [e_re_s]\pm[e_se_r]=0,$$

and substitute as above we get

$$(c) \qquad [e_re_s]\pm[e_se_r]=0.$$

Hence, by definition, (32) (e) and (f) give linear multiplications.

We have then four kinds of linear multiplication, viz:

1st. That in which there are no simplifying equations.

2nd. That in which all the products vanish.

3d. That whose law of simplification is $[e_ue_v]=[e_ve_u]$.

4th. That whose law of simplification is $[e_ue_v]=-[e_ve_u]$.

As between (e) and (f), the latter gives the simpler species of multiplication. To see this let us take the distributed product in (1) Art. 31. Equation (e) reduces the product, as we saw in that Article, to three terms. But taking (f) as the simplification law, we get a single term, viz., $(\alpha_1\beta_2-\alpha_2\beta_1)e_1e_2$.

34. The *Ausdehnungslehre* concerns itself very largely with the operation of multiplication, especially with what is called *combinatory* multiplication. This multiplication is based on the law described in the last Article, viz., $e_re_s=-e_se_r$, which also implies $e_r\,e_r=0$.

CHAPTER IV.

COMBINATORY MULTIPLICATION.

35. Definition. A product containing only units of the same system as factors and such that if the last two factors (called simple factors) are interchanged the sign of the product is changed is called *a combinatory product.*

Thus if E (not equal to 0) is a product of units, and e_1, e_2 are units, and

$$[Ee_1e_2]+[Ee_2e_1]=0,$$

the product $[Ee_1e_2]$ is a combinatory product.

36. *In the combinatory product $[Abc]$ in which A is any product of a series of factors and b and c are simple factors if b and c are interchanged, the sign of the product is changed.*

Proof. 1. Suppose b and c at first to be *units.* Since A is any series of factors and these factors are numerically derivable from the units, we may write, after removing the coefficients (by 28), $A=\Sigma\alpha_rE_r$, where E_r are products of the units. Substituting

$$[Abc]+[Acb]\equiv[\Sigma\alpha_rE_r.bc]+[\Sigma\alpha_rE_r.cb]=\Sigma\alpha_r[E_rbc]+\Sigma\alpha_r[E_rcb] \ (29)$$

$$=\Sigma\alpha_r\{[E_rbc]+[E_rcb]\} \ (16,\ 11)=0\ldots\ldots\ldots\ldots\ldots\ldots\ldots(35).$$

2. Next supposing b and c to be not units but numerically derivable from them. Let $b=\Sigma\beta_re_r$, $c=\Sigma\gamma_re_r$. Then

$$[Abc]+[Acb]\equiv[A.\Sigma\beta_re_r.\Sigma\gamma_re_r]+[A.\Sigma\gamma_re_r.\Sigma\beta_re_r]$$

$$=\Sigma\beta_r\gamma_s[Ae_re_s]+\Sigma\gamma_s\beta_r[Ae_se_r]\ldots\ldots\ldots\ldots\ldots(28)$$

$$=\Sigma\beta_r\gamma_s\{[Ae_re_s]+[Ae_se_r]\} \ (16)=0 \text{ (by 1 above)}.$$

37. *In a combinatory product one can interchange any two successive simple factors providing the sign of the product be changed, that is to say*

$$[AbcD]+[AcbD]=0,$$

where A and D are any factor series, and b and c are simple factors.

$$[Abcd]+[AcbD]=[\{Abc\}D]+[\{Acb\}D] \ (13.\ \text{Rem.})$$

$$=[(\{Abc\}+[Acb])D] \ (25)=0\ \ldots\ldots\ldots\ \ldots.(36).$$

38. *In a combinatory product one can interchange any two simple factors by changing the sign of the product.*

Thus, $P_{a,b}=-P_{b,a}$.

PROOF. Suppose n factors lie between a and b. Then n interchanges of adjacent factors will bring b into position next to a. After that $n+1$ interchanges of a with adjacent factors will put b in a's place and a in b's place. Thus there would be $2n+1$, or an odd number of changes of sign (37). Hence, ...

39. DEFINITION. If each of two series of quantities contain a and b once and but once and a stands *before* b in both or *after* b in both, then these quantities in those series are said to be *similarly arranged*; otherwise they are said to be *oppositely arranged.*

40. *Two combinatory products, which contain the same simple factors but in different order, are equal to each other or opposite in value according as the number of oppositely arranged pairs of factors is even or odd.*

Thus, $Q=(-1)^rP$, where P and Q are the two products and r is the number of oppositely arranged pairs of factors.

PROOF. If every pair of adjacent factors in Q were similarly arranged in P and Q, then, evidently, P and Q would be identical, and there would be no oppositely arranged pairs of factors in the two. If then there are oppositely arranged pairs of factors in P and Q, there must be at least one pair of factors adjacent in Q, which as compared with the same in P, is oppositely arranged. Suppose after this pair of factors is interchanged in Q we call the result Q_1. Then $Q_1=-Q$ (37). Evidently P and Q_1 will have one less pair of oppositely arranged factor pairs than P and Q. Thus if r was the number at first, Q_1 and P will have $r-1$ such pairs. If r is not 1, there must be another such factor pair in Q_1 and P. Repeating the operation we get $Q_2=(-1)^2Q$. If therefore there were r oppositely arranged factor pairs at first,

$$Q=(-1)^rP.$$

41. *If B is a combinatory product containing r factors and C one containing s factors, then*

$$[ABC]=(-1)^{rs}[ACB].$$

PROOF. Let $C=c_1c_2\ldots\, c_s$. Then since there will be a change of sign (37) each time c_1 interchanges with one of the r factors of B,

$$[ABc_1c_2\ldots.c_s]=(-1)^r[Ac_1Bc_2c_3\ldots.c_s]=(-1)^r(-1)^r[Ac_1c_2Bc_3\ldots.c_s]$$

$$=(-1)^{sr}[Ac_1c_2c_3\ldots.c_sB]=(-1)^{sr}[ACB].$$

42. *If A, B, C are products containing respectively, q, r, s factors, then from the preceding article it is plain that*

$$[ABC]=(-1)^{rs+sq+qr}[CBA].$$

43. *If two simple factors of a combinatory product are equal, the product is zero.*

PROOF. $P_{a,b}+P_{b,a}=0$ ....(38).
Then, if $b=a$, $P_{a,a}=0$.

44. *A combinatory product equals zero if a numerical relation exists between its simple factors.*

Let $a_1=\alpha_2 a_2+\alpha_3 a_3+\ldots\ldots\alpha_m a_m$. Then

$$[a_1 a_2\ldots\ldots a_m]\equiv[(\alpha_2 a_2+\alpha_3 a_3+\ldots\ldots\alpha_m a_m)a_2 a_3\ldots\ldots a_m]$$

$$=\alpha_2[a_2 a_2 a_3\ldots\ldots a_m]+\alpha_3[a_3 a_2 a_3\ldots\ldots a_m]+\ldots\ldots\ldots\ldots(29)$$

$$=0+0+\ldots\ldots\ldots\ldots\ldots\ldots(43).$$

45. *The combinatory product of n simple factors which are numerically derivable from the n quantities a_1, a_2, $\ldots\ldots a_n$ equals the determinant formed from the numerical coefficients in their values times $[a_1 a_2\ldots\ldots a_n]$.* Thus

$$[(\alpha_{11}a_1+\ldots\ldots\alpha_{n1}a_n)(\alpha_{12}a_1+\ldots\ldots\alpha_{n2}a_n)\ldots\ (\alpha_{1n}a_1+\ldots\ldots\alpha_{nn}a_n)]$$

$$=\begin{vmatrix}\alpha_{11} & \alpha_{12}\ldots\ldots\alpha_{1n}\\ \alpha_{21} & \alpha_{22}\ldots\ldots\alpha_{2n}\\ \ldots\ldots & \ldots\ldots\ldots\ldots\\ \ldots\ldots & \ldots\ldots\ldots\ldots\\ \alpha_{n1} & \ldots\ldots\ldots\ \alpha_{nn}\end{vmatrix}[a_1 a_2 a_3\ldots\ldots a_n].$$

For, the product on the left side equals by (28),

$$\Sigma\alpha_{s1}\alpha_{t2}\ldots\ldots\alpha_{wn}[a_s a_t\ldots\ldots a_w].$$

where the n subscripts s, t, $\ldots\ldots w$ assume in turn all the values from 1 to n. Now those terms of the distributed product in which two or more of the a's are equal disappear by (43). There remains then only those terms which contain all the a's used each once. We have now

$$[a_s a_t\ldots\ldots a_w]=(-1)^r[a_1 a_2 a_3\ldots\ldots a_n]\ldots\ldots\ldots\ldots\ldots(40),$$

where r denotes the number of pairs of factors in the left member oppositely arranged as compared with $[a_1 a_2 a_3\ldots\ldots a_n]$. But this is the law which determines the sign of any term of a determinant. See *e. g. Salmon's Higher Algebra*, §8.

46. DEFINITION. By the *multiplicative combinations* of a series of quantities are meant those products which are their combinations without repetition. The simple factors are called the elements of the combination.

47. *Every combinatory product of m factors which are numerically expressed in terms of the n independent quantities a_1, $a_2\ldots\ldots a_n$ is numerically expressible in terms of the multiplicative combinations of the mth class of $a_1\ldots\ldots a_n$, and each of these combinations has for its coefficient the determinant formed out of the m^2*

numerical coefficients belonging to its m elements. Thus

$$[\Sigma\alpha_a a_a . \Sigma\beta_b a_b \ldots.] = \Sigma \begin{vmatrix} \alpha_r & \cdots & \cdots \\ \vdots & \beta_s & \vdots \\ \vdots & & \ddots & \vdots \\ \vdots & & & \ddots \end{vmatrix} [a_r a_s \ldots.]$$

where $r < s < \ldots.$

PROOF. Evident from that of Art. 45.

48. *If $a_1 \ldots. a_n$ are independent, then their multiplicative combinations of any particular class are independent.*

PROOF. Let $\alpha A + \beta B + \ldots. = 0$, in which $A, B, \ldots$ are the multiplicative combinations of any one class formed out of $a_1 \ldots. a_n$, and $\alpha, \beta \ldots.$ are numbers. Let us multiply the equation through by A', the product of all the factors not found in A. Then $B, C, \ldots.$ would each contain one or more of the elements of A', and the products $[BA'], [CA'], \ldots.$ would each equal zero (43). Then we have $\alpha[AA'] = 0$. Now, $[AA']$ is not equal to zero. Hence $\alpha = 0$. In the same way we can prove that $\beta, \gamma, \ldots.$ must each equal zero. Hence there can be no such equation as $\alpha A + \beta B + \ldots = 0$, which expresses a dependence between $A, B, \ldots.$ Thus, $A, B, \ldots.$ are independent.

49. *A combinatory product remains constant when to any simple factor an arbitrary multiple of another is added.*

PROOF. $P_{a, b+qa} = P_{a, b} + qP_{a, a}$ (29) $= P_{a, b}$ (43).

50. DEFINITION. If from a series of quantities a second is derived by adding to any quantity a multiple of an adjacent quantity, then the first series is said to be changed into the second by a *simple linear alteration*. If the operation is repeated it is called a *multiple linear alteration*.

From what we saw in 49, it appears that the value of a quantity is not affected by linear alteration.

51. DEFINITION. The multiplicative combinations of the original units of the mth class is called a *unit* of the mth order, and a quantity numerically derived from such units is called a *quantity* of the mth order. A quantity of the mth order is said to be a *simple* quantity if it can be expressed as the combinatory product of m quantities of the first order; otherwise it is *complex*.

The space derived from the simple factors of a quantity (17) is called the space of this quantity. A quantity is subordinate to another if its space is.

52. DEFINITION. The outer product of two *units* of a higher order is obtained by merely uniting their simple factors into a combinatory product.

Thus, $[(e_1 e_2 \ldots e_m)(e_{m+1} \ldots. e_n)] = [e_1 e_2 \ldots e_n]$.

53. *In order to multiply two simple quantities, $[ab \ldots.]$ and $[cd \ldots.]$ it is sufficient to unite their simple factors taken in order into a single combinatory product $[ab \ldots. cd \ldots.]$.*

Proof.—Let $e_1 \ldots\ldots e_n$ be the original units, and let $a=\Sigma\alpha_a e_a$, $b=\Sigma\beta_b e_b$, $c=\Sigma\gamma_c e_c$, $d=\Sigma\delta_d e_d$. Then

$$[(ab\ldots\ldots)(cd\ldots\ldots)]\equiv[(\Sigma\alpha_a e_a.\Sigma\beta_b e_b\ldots\ldots)(\Sigma\gamma_c e_c.\Sigma\delta_d e_d\ \ldots\ldots)]$$

$$=[\Sigma\{\alpha_a\beta_b\ldots\ldots[e_a e_b]\}\Sigma\{\gamma_c\delta_d\ldots\ldots[e_c e_d]\}]\ldots\ldots\ldots\ldots(28)$$

$$=\Sigma\{\alpha_a\beta_b\ldots\ldots\gamma_c\delta_d\ldots\ldots[(e_a e_b\ldots\ldots)(e_c e_d\ldots\ldots)]\}\ \ldots(28)$$

$$=\Sigma\{\alpha_a\beta_b\ldots\ldots\gamma_c\delta_d\ldots\ldots[e_a e_b\ldots\ldots e_c e_d\ldots\ldots]\}\ldots\ldots(52)$$

$$=[\Sigma\alpha_a e_a.\Sigma\beta_b e_b\ldots\ldots\Sigma\gamma_c e_c.\Sigma\delta_d e_d\ldots\ldots]\ldots\ldots\ldots\ldots(28)$$

$$\equiv[ab\ldots\ldots cd\ldots\ldots].$$

54. Corollary to 53.—If a simple quantity A is subordinate to B (51), then B may be written $B=[AC]$, where C is a simple factor.

55. *To show that* $[A(BC)]=[ABC]$, *i. e. to show that the associative law* holds.

Proof.—1. When A, B, and C are the products of simple factors. The truth of this case follows readily from 53.

2. When A, B, and C are sums of simple quantities, $A=\Sigma A_a$, $B=\Sigma B_b$, $C=\Sigma C_c$.

$$[(A(BC)]\equiv[\Sigma A_a.(\Sigma B_b\Sigma C_c)]=\Sigma[A_a(B_bC_c)]\ldots\ldots\ldots\ldots(28)$$

$$=\Sigma[A_aB_bC_c]\text{ (by 1, above) }=[\Sigma A_a.\Sigma B_b.\Sigma C_c]\text{ (28) }=[ABC].$$

CHAPTER V.

RELATIVE MULTIPLICATION.

56. Definition.—In the preceding chapter no reference was made to the space in which the factors multiplied were contained. Now, in ordinary multiplication of geometrical magnitudes, there is a limit beyond which one can not go. For instance, when one has multiplied the length, breadth, and thickness together, he can add no other dimension. This suggests the idea of taking any arbitrary number as n for the number of dimensions of the space considered. In any investigation, then, what we will call "the space considered" is that space of the original units which contains all the quantities involved. Multiplications made with reference to the space considered are called *Relative Multiplications.*

57. Definition.—If in a space of the nth order, the combinatory product of the original units $e_1 e_2\ \ldots\ldots e_n$, is set equal to the scalar unity, and E is a unit of any order, (*i. e.* either one of the original units or a combinatory product of two or more of them), then the *complement of* E is $+E'$ or $-E'$, where E' is the combinatory product of all the units which do not appear in E. The complement of E is $+E'$ when $[EE']=+1$; and $-E'$ when $[EE']=-1$. Let the com-

plement of E be denoted by $|E$. This mark, the sign of the complement, in Grassmann is a vertical line somewhat longer that the caps and about as heavy as the vertical stroke in cap N. Then

$$|E=[EE']E'.$$

The end sought is to get $[E|E]=+1$. To show that this is attained, multiply the equation above through by E. Then

$$[E|E]=[EE'][EE']=+1,$$

since whether $[EE']=+1$ or -1, $[EE'][EE']=+1$.

The reason why we have this ambiguity of sign in the product $[EE']$ is because each original unit is allowed to have either the plus or minus sign.

In particular we have

$$|1=1, \text{ or } |\alpha=\alpha,$$

by multiplying the first equation through by α.

58. Definition.—By the *Complement* of any quantity A will be understood that quantity $|A$ which is obtained by replacing each product of units in the derived expression for A by its complement. Expressing the same in a formula we have

$$|A\equiv|(\alpha E_1+\alpha_2E_2+\ldots\ldots)=\alpha_1|E_1+\alpha_2|E_2+\ldots\ldots$$

where E_1, E_2, ... are products of units of any order.

59. *The complement of the complement of any quantity A is equal either to A, or to $-A$, according as $(-1)^{qr}$ is + or —*, where q and r are the orders of the quantity and its complement.

The Proof depends on 41.

60. *If the order of a space n is odd. $||A=A$: if n is even. $||A=(-1)^qA$. where q is the order of A.*

Proof.—By 59, $||A=(-1)^{q(n-q)}A$. Then if n is odd, $q(n-q)$ is even, whether q be even or odd : but if n is even, then if q is even $q(n-q)$ is even, and if q is odd $q(n-q)$ is odd. The theorem as stated readily follows.

61. Definition.—If the sum of the orders of two units is less than or equal to the order n of the space considered, then by their *progressive product* is understood their outer product (52), with the provision, however, that the product of the n original units is unity. On the other hand if the sum of the orders of two units is greater than the order n of the space, then by their *regressive product* is understood that quantity whose complement is the progressive product of the complements of these units.

Thus, if the sum of the orders of E and $F>n$, we have that

$$|[EF]=[|E|F],$$

where $[e_1 e_2 \ldots\ldots e_n]=1$.

The regressive products can be made plainer by an example. Let 5 be the order of the space considered, $[e_1 e_2 \ldots\ldots e_5]=1$, and let the product of $E_1=[e_1e_2e_3e_4]$ and $E_2=[e_1e_2e_5]$ be required.

Changing the order of the factors of E_2, we write $E_2=[e_5e_1e_2]$ (37), there being two interchanges. Then

$$[E_1E_2]=[e_1e_2e_3e_4][e_5e_1e_2]=[e_1e_2e_3e_4e_5e_1e_2] \text{ (52)}=[e_1e_2] \text{ (Rem. Art. 13)}.$$

Thus the product of E_1 and E_2 is the product of their common factors e_1 and e_2. The question arises, how can the common factors be selected and their product formed out of the two given factors?

Since E_1 and E_2 together contain all five of the units and neiter $|E_1$ nor $|E_2$ contains e_1 or e_2, the product of $|E_1$ and $|E_2$ contains all the units except e_1 and e_2. Then $|[E_1E_2]$ contains the factors of $[|E_1|E_2]$. Hence the definition of a regressive product.

62. *If q and r are the orders of A and B and n that of the space considered, the order of the product $[AB]$ is equal to $q+r$ when $q+r<n$, but is equal to $q+r-n$ when $q+r\geqq n$. In the language of the Theory of Numbers if p is the order of the product*

$$p\equiv q+r \text{ (Modulus } n).$$

The proof of this theorem follows the lines of the example of the preceding article.

63. *Similarly, for a larger number of factors,*

$$p\equiv q+r+s+t\ldots\ldots \text{ (Modulus } n).$$

64. *The product of the complements of two quantities is the complement of the product of those quantities, that is to say,*

$$[|A\ |B]=|[AB].$$

PROOF.—1. Suppose at first that the sum of the orders α and β of A and B is greater that n, that of the space under consideration. Let $A=\Sigma\alpha_r E_r$, $B=\Sigma\beta_s F_s$, where E_r and F_s are units. Then $|A=\Sigma\alpha_r|E_r$ and $|B=\Sigma\beta_s|F_s$ (58). Thus, we have.

$$[|A\ |B]\equiv[\Sigma\alpha_r|E_r.\Sigma\beta_s|F_s]=\Sigma\alpha_r\beta_s[|E_r|F_s \ldots\ldots\ldots\ldots\text{(28)}$$

$$=\Sigma\alpha_r\beta_s|[E_rF_s] \ \ldots\ldots\text{(61)}=|\Sigma\alpha_r\beta_s[E_rF_s] \ \ldots\ldots\text{(58)}$$

$$=|[\Sigma\alpha_rE_r.\Sigma\beta_sF_s]\ldots\ldots\text{(28)}\equiv|[AB].$$

2. Suppose $\alpha+\beta=n$. Let E and F be products of the original units. Two cases may be distinguished. First, when E and F contain a common factor e_1. It is plain that in this case both $[EF]$ and $[|E|F]$ contain common factors, so that they are each equal to zero (43). Second, when $[EF]=1$. Replacing $|E$ and $|F$ by their values from 57 and noting that $[FE][FE]=+1$, we get $[|E|F]=[EF]$. But as $[EF]=1$, $|[EF]=1$ (57). Thus the law holds for units. Then reasoning as in 1, above, it holds for any quantities.

3. Suppose $\alpha+\beta<n$. The proof of this case is based on 1. of this article by letting $A=|A'$, $B=|B'$, and writing

$$|[A'B']=[|A'\,|B']=[AB].$$

65. *The product of the complements of several quantities is the complement of the product of these quantities.*

Proved by mathematical induction from 64.

66. *The complement of a polynomial is the sum of the complements of its parts.*

Proof from 58.

67. *If E, F, G are units the sum of whose orders is n (the order of the space considered),*

$$[EF.EG]=[EFG]E.$$

PROOF.—We distinguish two cases: either $[EFG]$ contains equal factors or it does not. If it does, then at least one of the units, say e_1, is missing in it. Now let $[EF]=|Q$; then by 57 Q must contain e_1; likewise, let $[EG]=|R$; then R contains e_1. Then $[QR]=0$ (43). Now we have

$$[EF.EG]=[|Q|R]=|[QR]\ (64)=|0=0\ (57).$$

But $[EFG]$ also equals zero since it contains equal factors (43). Thus in this case

$$[EF.EG]=[EFG]E.$$

If $[EFG]$ does not contain equal factors, then it contains all of the original units and no others. Then by 57 and 55,

$$|G=[GEF][EF], \quad |F=[FEG][EG].$$

Since $[GEF]$ and $[FEG]$ are equal to either $+1$ or -1, they can appear on either side of their equations. Hence we can write

$$[EF]=[GEF]|G, \quad [EG]=[FEG]|F. \quad \text{Whence}$$

$$[EF.EG]=[GEF][FEG][|G|F]=[GEF][FEG]|[GF\ldots\ldots.(64)$$

$$=[GEF][FEG][GFE]E\ldots(57)=[GEF][EFG][GEF]E\ldots(41)$$

$$=[EFG]E,$$

since $[GEF][GEF]$, as in (57), equals $+1$.

68. *If A, B, C are simple quantities the sum of whose orders equals n, the order of the space of these quantities.*

$$[AB.AC]=[ABC]A.$$

The proof of this formula (based on (67) on account of its length, is omitted, as are also those of the next four formulas given below.

69.—71. *If A, B, C are simple quantities whose product is of the 0th order.*

69. $$[AB.AC]=[ABC]A.$$

70. $$[AB.BC]=[ABC]B.$$

71. $$[AC.BC]=[ABC]C.$$

72. *If A, B, C are simple quantities and the sum of the orders of A and C equals the order of the space considered and B is subordinate* (18) *to A, then*

$$[A.BC]=[AC]B \text{ and } [CB.A]=[CA]B.$$

Remark.—It seems proper to state here that the matter contained in Chapters II—V is taken direct from Grassmann's Ausdehnungslehre of 1862. What the writer has done has been to cut out everything which was not essential to the development of the main principles of the work. What to insert and what to omit constitutes the chief difficulty. In the following chapters (except Chapter VIII) we shall not follow Grassmann very closely.

CHAPTER VI.

GEOMETRICAL ADDITION AND SUBTRACTION.

73. The general theory of the *Ausdehnungslehre* may be applied in such diverse sciences as geometry, mechanics, and logic. We proceed in this and the following chapter to apply it in geometry.

74. The concepts dealt with in geometry are the *point*, *line*, *surface*, and *solid*, which may or may not be fixed in position. For the sake of distinction a line whose length and direction are fixed but not its position is called a *vector*. (3). A portion of a plane whose direction and extent are fixed but not the position of the plane is called by analogy a *plane vector*.

75. As an introduction to the *Ausdehnungslehre* the addition and subtraction of vectors was treated in Chapter I. It is evident from what was given in that chapter that plane vectors may be added and subtracted in the same way as

line vectors. One gets the parts whose sum is a given plane vector by projecting the given plane vector on the coördinate planes.

As we have already treated of the addition and subtraction of vectors, we proceed to apply the laws of addition and subtraction to points.

76. We will define a point as an infinitesimal portion of a line and denote it by p. When the point has position we will denote it by $p\rho$, in which p denotes the point at the extremity of the radius vector ρ from the origin, O. Evidently a line, or plane, or solid may be located by means of a radius vector in the same way. (See 168.)

Grassmann defines a point as that which has position and uses a single letter as A to denote it. In what follows if the ρ's be cut out of the formulas and the p's be given the subscripts of the ρ's, Grassmann's expressions will result.

The reasons for using the complex symbol $p\rho$, instead of the simpler A, are: (1) Because the concept is complex and therefore for clearness should be represented by a complex symbol. (2) Since position is relative, for the proper representation of positions an origin is needed. (3) Because this notation shows plainly the relation which exists between point and vector analysis.

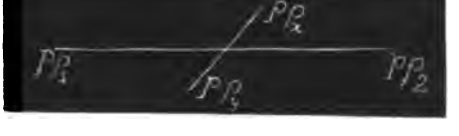
77. What we will call *unit* points all have the same (infinitesimal) unit length. This length as also that of the radius vector may be multiplied by any scalar, m. Thus $mp\rho$ denotes the point whose length or "weight," as it is called, is mp held in position by ρ, while $p(m\rho)$ denotes p held in position by $m\rho$, *i. e.*, m times the length of ρ.

78. The difference between two unit points, since they can differ only in position, is a certain distance in a certain direction, *i. e.*, is a vector. (See 3.)

Thus $\qquad p\rho_2 - p\rho_1 = \rho_2 - \rho_1 = \epsilon$. (4).

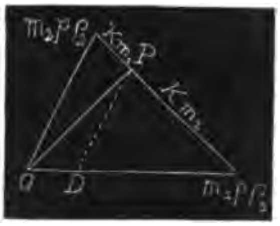
Similarly, $\quad mp\rho_2 - mp\rho_1 = m\epsilon$.

79. We next seek to find the sum of two or more points. What this sum is remains to be determined. Grassmann gives an investigation to show that the sum of two unit points is a point on the line joining them. We abridge this proof as follows: He begins by postulating (1) That whatever is true of one set of points is true of any congruent system wherever situated; (2) That the fundamental laws of addition and subtraction (14) hold. Then he assumes that the sum of two points is some point.



Let, in the figure, $p\rho_1$ and $p\rho_2$ be any two unit points whose sum is sought. Suppose $p\rho_1 + p\rho_2 = p\rho_x$. Then revolving the whole figure in the plane of the paper through 180°, $p\rho_1$ coincides with $p\rho_2$, $p\rho_x$ with $p\rho_y$, and $p\rho_2$ with $p\rho_1$. Thus we get $p\rho_2 + p\rho_1 = p\rho_y$. But by 14, $p\rho_2 + p\rho_1 = p\rho_1 + p\rho_2$. Then $p\rho_x = p\rho_y$. This can only happen when they both coincide with the midpoint of the straight line joining the two given points.

80. Mechanics gives us a simpler and more general interpretation for the sum of two or more points. Let us regard the points as parallel (infinitesimal) forces whose magnitudes are represented by the weights of the points. The law for the addition of parallel forces gives a simple and consistent result. Thus



$$m_1 p\rho_1 + m_2 p\rho_2 = (m_1 + m_2) p \left(\frac{m_1\rho_1 + m_2\rho_2}{m_1 + m_2} \right),$$

i. e., the sum of the two weighted points is a point on the line joining them whose weight is the sum of the weights of the two points and the extremity of whose radius vector divides the line joining the two given points into segments inversely proportional to the weights of these points.

That $\frac{m_1\rho_1 + m_2\rho_2}{m_1 + m_2}$ is the vector to the point described is evident from elementary geometry. Thus regarding the two radii vectores going out from O as axes, it is easy to show by similar triangles that

$$OD = \frac{m_1}{m_1 + m_2}\rho_1, \quad \text{and} \quad DP = \frac{m_2}{m_1 + m_2}\rho_2.$$

NOTE.—The letter k which appears on the figure above is a factor chosen such that km_1 and km_2 equal the segments designated by them.

81. *Generalizing the result of the last article, we have*

$$\Sigma m_1 p\rho_1 = \Sigma m_1 . p \left(\frac{\Sigma m_1\rho_1}{\Sigma m_1} \right).$$

82. When $\Sigma m_1 = 0$ in the preceding result, the weight of the sum point is zero and the radius vector is infinite in length. To interpret this, we get the sum of all the points except one and then add this partial sum to the remaining point. In this way we obtain an expression similar to that of 78 where the result is a *vector*.

Hence $\Sigma m_1 p\rho_1$ *is a VECTOR when* $\Sigma m_1 = 0$, *and a POINT when* Σm_1 is not equal to 0.

Thus a point at infinity (of zero weight) is equivalent to a vector.

83. Using the formula of 80 and putting

$$(m_1 + m_1) = -m_3 \text{ and } \frac{m_1\rho_1 + m_2\rho_2}{m_1 + m_2} = \rho_3,$$

we see that $m_1 p\rho_1 + m_2 p\rho_2 + m_3 p\rho_3 = 0$, and $m_1 + m_2 + m_3 = 0$, *are the conditions that the three points* $p\rho_1$, $p\rho_2$, $p\rho_3$ *shall be collinear and the the vectors* ρ_1, ρ_2, ρ_3, *coplanar.*

84. For space of three dimensions we have (82)

$$m_1p\rho_1+m_2p\rho_2+m_3p\rho_3+m_4p\rho_4=0, \text{ and } m_1+m_2+m_3+m_4=0,$$

for the conditions that the four points $p\rho_1$, $p\rho_2$, $p\rho_3$, $p\rho_4$ shall be coplanar.

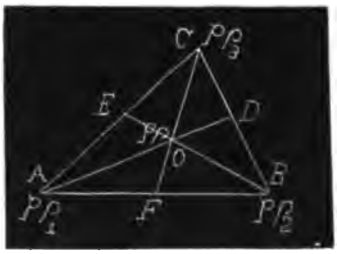
85. *From the equations of* 80, 83, 84 *we see that two points are independent, three or more collinear points are dependent* (10), *three non-collinear points are independent, four or more coplaner points are dependent, four non-coplaner points are independent, and any five or more points are dependent in solid space.*

86. The calculus of this chapter is evidently adapted to dealing with theorems concerning the collinearity of points in geometry, and the center of parallel forces in mechanics.

87. We conclude this chapter with an example. *Required to find whether the three medians of a triangle meet in a point.*

Let the vertices of a triangle ABC be located by the unit points $p\rho_1$, $p\rho_2$, $p\rho_3$, and D, E, F be the mid-points of the sides. We have then

$$D=\frac{p\rho_2+p\rho_3}{2},\ E=\frac{p\rho_1+p\rho_3}{2}.$$



If $p\rho$ denote a unit point at O the intersection of AD and BE, and x, y, x', and y' arbitrary scalars, we may write

$$p\rho=xp\rho_1+y\frac{p\rho_2+p\rho_3}{2}=x'p\rho_2+y'\,\frac{p\rho_1+p\rho_3}{2}.$$

Then (20), $x=\frac{1}{2}y'$, $y=y'$; whence $x=\frac{1}{2}y$. But $x+y=1$ (77). Then $x=\frac{1}{3}$, $y=\frac{2}{3}$. Hence

$$p\rho=\tfrac{1}{3}p\rho_1+\tfrac{2}{3}\left(\frac{p\rho_2+p\rho_3}{2}\right)=\tfrac{1}{3}p\rho_1+\tfrac{1}{3}p\rho_2+\tfrac{1}{3}p\rho_3.$$

By symmetry we see that the intersection of AD and CF must be the same point. Or, supposing O to be the intersection of BE and CF, we may test A, O, and D for collinearity directly.

$$\begin{array}{ccc} A & O & D \\ \tfrac{1}{3}p\rho_1- & 1\,(\tfrac{1}{3}p\rho_1+\tfrac{1}{3}p\rho_2+\tfrac{1}{3}p\rho_3)+\tfrac{2}{3}(\tfrac{1}{2}(p\rho_2+p\rho_3))\equiv 0. & (83). \end{array}$$

It is evident that the above equations can be interpreted as equations of ordinary vector analysis by dropping the p's. In this way is shown the relation existing between point and vector analysis.

CHAPTER VII.

GEOMETRICAL MULTIPLICATION.

88. Basing our investigation on the fundamental law of combinatory multiplication (34), *let us seek the product of a non-posited point* (76) *and two vectors.* The vectors are thought of as denoting merely translation a given distance in a given direction (See 4—9). Let p denote the point and α and β the vectors.

Suppose

$$\alpha=x_1\varepsilon_1+y_1\varepsilon_2$$
$$\beta=x_2\varepsilon_1+y_2\varepsilon_2$$

where ε_1 and ε_2 are unit vectors at right angles to each other. Then by 45

$$[p\alpha\beta]=\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}[p\varepsilon_1\varepsilon_2].$$

Now the determinant $x_1y_2-x_2y_1$ is the difference between two rectangles. Let us seek the relation, if any exists, between this area and that of the parallelogram $AOCB$. We have

$$OACB=OBCE-OACE=BCED-OAFE$$
$$=\tfrac{1}{2}(y_1+2y_2)x_1-\tfrac{1}{2}(x_1+2x_2)y_1=x_1y_2-x_2y_1.$$

The equation $[p\alpha\beta]=(x_1y_2-x_2y_1)[p\varepsilon_1\varepsilon_2]$ shows, therefore, that if $p\varepsilon_1\varepsilon_2$ is taken to denote the area of the unit square the two sides of which are ε_1 and ε_2, $p\alpha\beta$ denotes the area of the parallelogram whose adjacent sides are α and β.

Now to assume that $p\varepsilon_1\varepsilon_2$ is the area of the square is a perfectly natural assumption analogous to the theorem in geometry which says, The area of a rectangle equals the product of the base by the altitude. *Thus Grassman was led to define the product $\alpha\beta$ as the area generated while α moves (remaining, of course, constantly parallel to itself) over a distance determined by β.*

Attaching this meaning to $p\varepsilon_1\varepsilon_2$ we think first of the point p moving over a distance determined by ε_1 generating a line, and then this line moving over a distance determined by ε_2 generating the square.

In the next two articles we will drop the factor p and think of the first factor as a line rather than as mere translation.

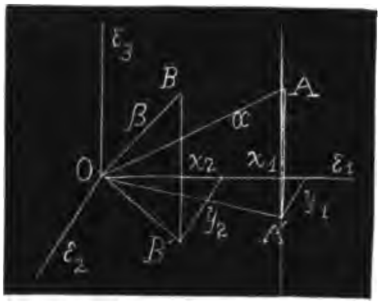
89. *Similarly let us seek the product of two vectors in space.* Let

$$\alpha=x_1\varepsilon_1+y_1\varepsilon_2+z_1\varepsilon_3 \text{ and } \beta=x_2\varepsilon_1+y_2\varepsilon_2+z_2\varepsilon_3$$

where ε_1, ε_2, ε_3 are three unit vectors each at right angles to the other two. Then by 47,

$$[\alpha\beta]=\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}[\varepsilon_1\ \varepsilon_2]+\begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix}[\varepsilon_2\ \varepsilon_3]+\begin{vmatrix} z_1 & x_1 \\ z_2 & x_2 \end{vmatrix}[\varepsilon_3\varepsilon_1].$$

Here it is evident that the first determinant coefficient is the area of the projection of the parallelogram whose adjacent sides are α and β on the plane $\varepsilon_1\varepsilon_2$, and that the other coefficients are the areas of the corresponding projections on the other planes. The above equation expresses then, that the area of the parallelogram whose sides are α and β is equal to the sum of its projections on the three coördinate planes (75).



90. *To find the product of three vectors, α, β, and γ.*

If ε_1, ε_2, ε_3 are three mutually perpendicular vectors, and

$$\alpha=x_1\varepsilon_1+y_1\varepsilon_2+z_1\varepsilon_3,\ \beta=x_2\varepsilon_1+y_2\varepsilon_2+z_2\varepsilon_3,\ \gamma=x_3\varepsilon_1+y_3\varepsilon_2+z_3\varepsilon_3,$$

$$[\alpha\beta\gamma]=\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}[\varepsilon_1\varepsilon_2\varepsilon_3] \qquad (45).$$

Hence if $[\varepsilon_1\varepsilon_2\varepsilon_3]$ is the volume of a cube each side of which is a linear unit, $[\alpha\beta\gamma]$ devotes the volume of the parallelopiped whose adjacent edges are α, β, γ, since, as is well known in analytic geometry, the determinant expresses the number of units of volume in the parallelopiped. In the *Ausdehnungslehre* attention must be paid to the *order* of the factors, *i. e.* to the order of generation. Thus (37) $[\alpha\beta]=-[\beta\alpha]$, and $[\alpha\beta\gamma]=-[\alpha\gamma\beta]$.

It is apparent from the preceding articles that the *Ausdehnungslehre* is especially well adapted for the investigation of propositions concerning the areas and volumes of rectilinear figures.

91. *To find the product of a posited point and vector.*

Let $p\rho$ and ε denote the point and vector. Following the areal interpretation given above this product should be a *line* whose length is ε and whose other dimension is the infinitesimal p, the whole fixed in position by the radius vector ρ.

Now $p(\rho+x\varepsilon)$ denotes any point on the line through $p\rho$. Then since

$$[p(\rho+x\varepsilon)\varepsilon]=[p\rho\varepsilon]+x[p\varepsilon\varepsilon]=[p\rho\varepsilon] \text{ (34)},$$

we see that the product of a posited point and a vector determine a line segment, *but this line segment may have any position on the vector through the given point.*

92. *To find the product of two posited points.*

Let $p\rho_1$ and $p\rho_2$ be two unit points. Then

$$[p\rho_1 . p\rho_2]=[p\rho_1(p\rho_2-p\rho_1)], \text{ since } [p\rho_1 . p\rho_1]=0. \quad (34)$$

But $[p\rho_1(p\rho_2-p\rho_1)]$ is the product of a point and a vector (78) which, by 91, is the line from the extremity of ρ_1 to that of ρ_2. Thus, *The product of two posited points is the line joining the first to the second.*

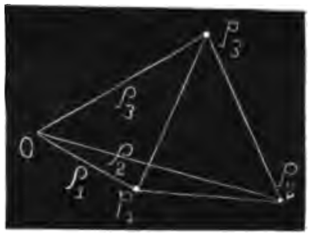
93. We have illustrated in the last article a principle which will be found to hold generally in the *Ausdehnungslehre*, viz., *That the product of posited quantities which have no common figure is some multiple of the connecting figure.*

94. *To find the product of three posited points.*

Let us use p_1, p_2, p_3 to denote the three unit points instead of $p\rho_1$, $p\rho_2$, $p\rho_3$ as heretofore. It will be understood when p is used to denote the posited point $p\rho$, that it stands for the complex quantity described in 76.

$$[p_1p_2p_3]=[p_1p_2.p_3] \text{ (Rem. 13) } =[p_1p_2(p_3-p_1)], \text{ since } [p_1p_2p_1]=0 \text{ by 43.}$$

But $[p_1p_2]$ is the line from p_1 to p_2 (92), and by 88 the product of a line $[p_1p_2]$ and a vector p_3-p_1 (78) equals the parallelogram whose adjacent sides are $[p_1p_2]$ and $[p_1p_3]$.



Thus the product of three given posited points is twice the area of the triangle whose vertices are the three given points. (93)

Let $p_1+xp_2+yp_3$ denote any point in the plane of $[p_1p_2p_3]$.

Now $[p_1p_2p_3]$=twice area of triangle whose vertices are p_1, p_2, p_3; but we have

$$[(p_1+xp_2+yp_3)(p_2-p_1)(p_3-p_1)]=[p_1p_2p_3] \quad (22, 43).$$

This shows that *the value of the product remains the same whatever be the position of the triangle $[p_1p_2p_3]$ in the plane of these points.*

95. *To find the product of four posited points.*

Let p_1, p_2, p_3, p_4, represent four unit points. Then

$$[p_1p_2p_3p_4]=[p_1(p_2-p_1)(p_3-p_1)(p_4-p_1)] \quad (43)$$
$$=6\times\text{tetraedron whose vertices are } p_1, p_2, p_3, p_4, \text{ by 90.}$$

Let $p_1+xp_2+yp_3+zp_4$ be any point whatever. Then

$$[(p_1+xp_2+yp_3+zp_4)(p_2-p_1)(p_3-p_1)(p_4-p_1)]=[p_1p_2p_3p_4]. \quad (22, 43)$$

Hence the product of four points in solid space is the same no matter where located.

96. We have used the terms "line" and "line segment" to denote a quantity whose length and the line in which it must lie are given but not its position in that line. Similarly we will use the terms "plane" and "plane segment" (See 74) to denote the corresponding areal quantity described in 94. Grassmann's terms for them are respectively "*Linientheil*" and "*Flächentheil.*" For the quantity described in 95 he uses the term "*Körpertheil.*"

97. *To find the sum or difference of two lines or two planes.*

Let $[p_1p_2]$, $[p_1p_3]$ be the lines, $[p_1p_2p_3]$, $[p_1p_2p_4]$, the planes, and let $p_3+p_4=2p_s$, and $p_3-p_4=\varepsilon$. Then

$$[p_1p_2]\pm[p_1p_3]=[p_1(p_2\pm p_3)]=2[p_1p_s], \text{ or } [p_1\varepsilon], \text{ a line.} \quad (82, 91)$$

$$[p_1p_2p_3]\pm[p_1p_2p_4]=[p_1p_2(p_3\pm p_4)]=2[p_1p_2p_s], \text{ or } [p_1p_2\varepsilon], \text{ a plane.} \quad (94)$$

98. *To find the sum of the sides of a triangle.*

Let $p\rho_1$, $p\rho_2$, $p\rho_3$ represent the three vertices of a triangle. Then

$$[p\rho_1\cdot p\rho_2]+[p\rho_2\cdot p\rho_3]+[p\rho_3\cdot p\rho_1]=[(p\rho_2-p\rho_1)(p\rho_3-p\rho_1)], \text{ since } [p\rho_1 p\rho_2]=0.$$

Thus the sum of the three sides of a triangle equals the product of the vectors $p\rho_2-p\rho_1$ and $p\rho_3-p\rho_1$. This product differs from the expression for the area of the triangle (94) *by the absence of the first factor* $p\rho_1$. An interpretation of the expression given above for the sum of the sides which makes it equal to the area of the triangle may be had by thinking of $p\rho$ as a *generative* product of p and ρ. Using the period to denote generative multiplication, we have

$$p.\rho_1.p\rho_2=OAB\ ;\ p.\rho_2.p\rho_3=OBC\ ;\ p.\rho_3.p\rho_1=-OCA.$$

$$\text{Thus, } OAB+OBC-OCA=p.\rho_1.p.\rho_2+p.\rho_3.p\rho_2+p.\rho_2.p\rho_3=ABC.$$

Remark.—By Grassmann's formulas the sum of the sides of a triangle equals its area, for he treats a point as that which has position only, and considers that the product of two vectors alone equals the area described in 88. The use of the definition of a point given in 76 has the effect of making some of the theorems of this chapter depart in certain respects from Grassmann's. The writer thinks however that regarding a line as generated by a point in motion agrees well with Grassmann's conception of "generative" multiplication.

99. *If* p_1, p_2, p_3, p_4, p_5, p_6 *are six points,* ε_1, ε_2, ε_3, ε_4 *are four vectors and* x *is any scalar, by the preceding articles we have the following conditions:*

(1) $p_1=xp_2$ is the condition that points p_1 and p_2 coincide.

(2) $[p_1p_2]=x[p_3p_4]$, is the condition that the (unlimited) lines $[p_1p_2]$ and $[p_3p_4]$ coincide.

(3) $[p_1p_2p_3]=x[p_4p_5p_6]$ is the condition that the (unbounded) planes $[p_1p_2p_3]$ and $[p_4p_5p_6]$ coincide.

(4) $\varepsilon_1=x\varepsilon_2$ is the condition that the vectors ε_1 and ε_2 are parallel.

(5) $[\varepsilon_1\varepsilon_2]=x[\varepsilon_3\varepsilon_4]$ is the condition that the planes of $[\varepsilon_1\varepsilon_2]$ and $[\varepsilon_3\varepsilon_4]$ are parallel.

100. A point is regarded as a space of the first order; an unlimited line as a space of the second order ; an unbounded plane as a space of the third order ; and solid space as a space of the fourth order. See 17 and 85. Since a vector may be regarded as a point at infinity, a vector also may be regarded as a quantity of the first order. See Chapter I.

101. Relative Products. A *Planimetric* product is a relative product whose factors are in a plane, or space of the third order. A Stereometric product is one whose factors are in a space of the fourth order (100).

102. *To find the planimetric product of two line segments* p_1p_2 and p_1p_3. We have

$$[p_1p_2\cdot p_1p_3]=[p_1p_2p_3]p_1.\ (67)$$

Here $[p_1p_2p_3]$ is a scalar (101, 61). Thus the product is the point of intersection multiplied by a scalar.

103. *To find the planimetric product of two parallel line segments* $[p_1p_2]$ *and* $[p_3p_4]$.

We have, by hypothesis, $p_3-p_4=x(p_1-p_2)$ (99, (4)). Then

$$[p_1p_2\cdot p_3p_4]=[(p_1-p_2)p_2\cdot(p_3-p_4)p_4]\ (49)=x[(p_1-p_2)p_2\cdot(p_1-p_2)p_4]\ \text{(Hyp.)}$$

$$=x[(p_1-p_2)p_2p_4](p_1-p_2)\ (67)=[(p_3-p_4)p_2p_4](p_1-p_2)\ \text{(Hyp.)}$$

$$=[p_3p_2p_4](p_1-p_2)\ (49)=[p_2p_3p_4](p_2-p_1).\ (38)$$

Hence the product is the point at infinity (the vector p_2-p_1) which is the intersection of the two lines multiplied by the scalar $[p_2p_3p_4]$.

104. The last two articles illustrate a principle of general application in the *Ausdehnungslehre*, viz., *That the relative product of posited quantities which have a common figure is that common figure multiplied by a scalar.* See 93.

105. *To find the planimetric product of two lines and a posited point.*

Let $[p_1p_2]$ and $[p_1p_3]$ be the lines and p the point. Then

$$[p_1p_2\cdot p_1p_3\cdot p]=[(p_1p_2p_3)p_1\cdot p]\ (13,\ \text{Rem.},\ 67)=[p_1p_2p_3][p_1p]$$

since $[p_1p_2p_3]$ is a scalar. (101, 61)

106. *To find the planimetric product of the three line segments* $[p_1p_2]$, $[p_1p_3]$, $[p_4p_5]$.

We have $[p_1p_2 \cdot p_1p_3 \cdot p_4p_5] = [p_1p_2p_3][p_1 \cdot p_4p_5]$ (67) $= [p_1p_2p_3][p_1p_4p_5]$ (55).

COROLLARY. *If the three line segments are the sides of a triangle we may write* $[p_2p_3]$ *instead of* $[p_4p_5]$. *Then the product is* $[p_1p_2p_3]^2$.

107. The following general principle is illustrated in the two preceding articles: *If at any time the product of factors combined in regular order from the left gives rise to a scalar or to a scalar times an extensive quantity, this scalar is to be regarded as a simple numerical factor, and the extensive quantity part of the product, if there is such, is to be combined with the remaining extensive factors, and so on.* Such products are described as *mixed, i. e.* as both progressive and regressive. (61)

108. *The stereometric products of a line and a point and of two lines are commutative; but that of a point and a plane is non-commutative.*

Let L_1 denote the line $[p_1p_2]$, L_2 the line $[p_3p_4]$, and P the plane $[p_2p_3p_4]$.

$$[L_1p_3] \doteq [p_1p_2 \cdot p_3] = [p_1p_2p_3] = [p_3 \cdot p_1p_2] \ (40,\ 55) \equiv [p_3L_1];$$

$$[L_1L_2] \doteq [p_1p_2 \cdot p_3p_4] = [p_1p_2p_3p_4] \ (55) = [p_3p_4 \cdot p_1p_2] \ (40,\ 55) \equiv [L_1L_2];$$

$$[Pp_4] \doteq [p_1p_2p_3p_4] = -[p_4 \cdot p_1p_2p_3] \ (40,\ 55) \equiv -[p_4P].$$

109. *To find the stereometric product of two non-incident plane segments* $[p_1p_2p_3]$ and $[p_1p_2p_4]$.

We have $[p_1p_2p_3 \cdot p_1p_2p_4] = [p_1p_2p_3p_4][p_1p_2]$ (67). See 104.

110. *To find the stereometric product of three plane segments* $[p_1p_2p_3]$, $[p_1p_2p_4]$, $[p_1p_3p_4]$ *which intersect in* p_1.

$$[p_1p_2p_3 \cdot p_1p_2p_4 \cdot p_1p_3p_4] = [p_1p_2p_3p_4][p_1p_2 \cdot p_1p_3p_4] \ (67) = [p_1p_2p_3p_4]^2p_1 \ (67).$$

111. *To find the stereometric product of a plane segment* $[p_1p_2p_3]$ *and line segment* $[p_1p_4]$ *which do not lie in the same plane. Is the product commutative?*

We have $[p_1p_2p_3 \cdot p_1p_4] = [p_1p_2p_3p_4]p_1$ (67) (See 104).

Also, $[p_1p_4 \cdot p_1p_2p_3] = [p_1p_4p_2p_3]p_1$ (67) $= [p_1p_2p_3p_4]p_1$ (40).

112. *The stereometric product of two line segments* $[p_1p_2]$ *and* $[p_3p_4]$ *equals zero when and only when they lie in the same plane* (55, 95); *the stereometric product of two quantities of the first, second, or third orders, but not both at the same time of the second orders, equals zero when and only when the quantities are incident, i. e. when one falls in the space of the other; as two coincident points, two plane segments if the planes coincide, a point and a line- or a plane-segment if the point lies in the line or plane, a line segment and a plane segment if the line lies in the plane.*

113. *Algebraic Curves and Surfaces. The equation of a variable point* p *which lies in the same straight line as* $[p_1p_2]$ *is* $[p\,p_1p_2] = 0$. (112)

114. *The equation of a straight line* L *which passes through the intersection*

of L_1 and L_2 and lies in their plane is $[L_1 L_2 L]=0$ where $[L_1 L_2 L]$ is a planimetric product (See 106).

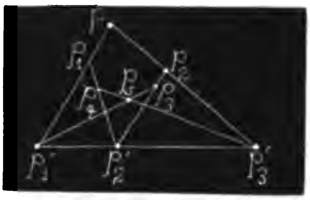
115. *If $P_{n,p}$ is a planimetric product of order zero which contains the variable point p, n times and besides only constant points and lines as factors, then $P_{n,p}=0$ is the point equation of an algebraic curve of the nth order, that is to say, the point p moves in an algebraic curve of the nth order.*

Proof. Let p_1, p_2, p_3 be any three points in the plane. Then

$$p=x_1 p_1+x_2 p_2+x_3 p_3$$

may be any point in the plane. Substituting this value of p in $P_{n,p}=0$ there results a homogeneous equation of the nth degree in x_1, x_2, x_3 whose terms are all of the form $Ax^a{}_1 x^b{}_2 x^c{}_3$ where $a+b+c=n$. A is the product of constant lines and points, and since this product is by hypothesis of the zeroth order, A is a constant. Regarding x_1, x_2, x_3 as trilinear coördinates, we see that $P_{n,p}=0$ now becomes an ordinary cartesian equation for a curve of the nth order.

116. As an example we give a proof of Pascal's Hexagram Theorem.



Let $p_1 \ldots p_5$ be five given points and p a variable point which moves so as to leave p'_2 on the line $[p'_1 p'_3]$, p'_1, p'_2, p'_3 being defined by the following equations: $p'_1=[pp_1 \cdot p_3 p_5]$; $p'_2=[p_2 p_3 \cdot p_1 p_4]$; $p'_3=[pp_2 \cdot p_4 p_5]$. $[(pp_1 \cdot p_3 p_5)(p_2 p_3 \cdot p_1 p_4)(pp_2 \cdot p_4 p_5)]=0$ is the equation of a conic passing through the five given points. For it is of the second degree in p and is satisfied by putting p equal to any one of the five given points. By changing points into lines in the above we have Brianchon's Theorem.

CHAPTER VIII.

INNER PRODUCTS,—NORMAL SYSTEMS,—PROJECTION.

117. Definition.—The Inner Product of two *units* of any order is the relative product of the first and complement of the second.

Thus the inner product of E and F is $[E|F]$.

Note.—Grassmann seems to have regarded the outer (52) and inner products as different in nature. But they both obey the laws of combinatory multiplication, the complement sign indicating a preliminary change to be made in the factor following it before it is combined with the other.

118. *The inner product of any two quantities is equal to the relative product of the first and complement of the second.*

Proof.—Let $A=\alpha A_1+\ldots+\alpha_n A_n$, $B=\beta B_1+\ldots+\beta_n B_n$, where $A_1, \ldots$, $B_1, \ldots$, are units. Also for the moment let $\times$ signify the inner product.

Then $[A \times B]\equiv[(\alpha_1 A_1+\ldots+\alpha_n B_n)\times(\beta_1 B_1+\ldots\beta_n B_n)]$

$$=\Sigma \alpha_r \beta_s [A_r \times B_s]. \quad (28)$$

Now since $A_1, \ldots, B_1, \ldots$, are *units*, $[A_r \times B_s] = [A_r \mid B_s]$. (117)

$$\text{Then } [A \times B] = \Sigma\alpha_r\beta_s[A_r \mid B_s] = \Sigma[\alpha_r A_r . \Sigma\beta_s \mid B_s] \ (28)$$

$$\equiv [A\Sigma\beta_s \mid B_s] = [A \mid \Sigma\beta_s B_s] \ (58) \equiv [A \mid B].$$

119. *The inner product of two quantities of the same order is a number.* For, letting r denote the order of each factor, the complement of the second factor is of order n-r, and the product of the first factor which is of the order r and another which is of order n-r is of the nth order, *i. e.* is a pure number. (61)

COROLLARY.—*On account of the scalar value of the product, in this case* $[A \mid B] = [B \mid A]$.

120. *The inner product of two equal units is unity, while that of two different units of the same order is zero.*

Thus $[E_1 \mid E_1] = 1$ (57), $[E_r \mid E_s] = 0$. (43)

121. *If* $E_1, \ldots, E_n$ *are units of any order, but all of the same order, then*

$$[A \mid B] \equiv [(\alpha_1 E_1 + \ldots + \alpha_n E_n) \mid (\beta_1 E_1 + \ldots + \beta_n E_n)]$$

$$= \alpha_1\beta_1 + \alpha_2\beta_2 + \ldots + \alpha_n\beta_n. \ (120)$$

122. *If* $B=A$ *in* 121, *we get what is called* the inner square of A, which is denoted by A^{2}; thus we have

$$A^{2} = \alpha_1^2 + \alpha_2^2 + \ldots \alpha_n^2.$$

123. NORMAL SYSTEMS.—DEFINITION.—The numerical value of an extensive quantity A is defined as the positive square root of the inner square of A. This definition reminds one of the modulus in complex numbers.

124. DEFINITION.—Two quantities (which do not equal zero) are said to be normal to each other if their inner product is zero. Two spaces are said to be every way (allseitig) normal to each other when each quantity in either space is normal to every quantity in the other space.

125. DEFINITION.—A normal system of the nth order is a set of n numerically equal quantities of the first order of which each is normal to every other. If at the same time n is the order of the space, then such quantities constitute a *perfect* normal system. The numerical value of these n quantities is at the same time the numerical value of the system. Every normal system whose numerical value is unity is called a *simple* system.

126. DEFINITION.—By Circular Alteration is meant that transformation of a system by which two quantities a and b of the system are transformed respectively into $xa+yb$ and $\pm(xb-ya)$, where $x^2+y^2=1$. The circular alteration is said to be positive or negative according as $+$ or $-$ is taken in the double sign.

127. *By circular alteration any normal system is transformed into another normal system having the same numerical value.*

PROOF.—Suppose $a, b, \dots$ to be the quantities of a normal system. Then, by definition,

$$0=[a \mid b]=[a \mid c]=[b \mid c]=\dots, \text{ and } a^2=b^2=c^2=\dots$$

Let now a change into $a_1=xa+yb$ and b into $b_1=\pm(xb-ya)$ where $x^2+y^2=1$. We are to show that $a_1, b_1, c \dots$ constitute a normal system. We have

$$a_1{}^2=(xa+yb)^2=x^2a^2+y^2b^2, \text{ since } [a \mid b]=0,$$

$$=(x^2+y^2)a^2=a^2, \text{ by hypothesis.}$$

Similarly, we can prove $b_1{}^2=b^2$.

Also, $[a_1 \mid b_1]=\pm[(xa+yb) \mid (xb-ya)]=\pm xy(b^2-a^2)=0$.

Finally, $[a_1 \mid c]=[(xa+yb)c]=x[a \mid c]+y[b \mid c]=0$.

Hence, by definition, $a_1, b_1, c, \dots$ constitute a normal system.

128. *The combinatory product of quantities of a normal system is unaltered by positive circular alteration, and has its sign changed by negative circular alteration.*

Using the notation of 127, we have

$$[a_1b_1]=[(xa+yb)(xb-ya)]=x^2[ab]-y^2[ba] \ (34)=(x^2+y^2)[ab]=[ab].$$

129. *All the quantities of a normal system are independent.*

PROOF.—Suppose $a, b, c, \dots$ to be quantities of a normal system. Let us assume for the moment that they are not independent and that

$$a=\beta b+\gamma c+\dots$$

We multiply both sides by $|a$. Then

$$a^2=\beta[b \mid a]+\gamma[c \mid a]+\dots=0. \ (124)$$

But $a^2=0$ contradicts the hypothesis in 124. Hence the quantities of a normal system are independent.

130. *The system of the original units* (11) *is a perfect normal system whose numerical value is unity* (125).

PROOF.—Let $e_1, \dots e_n$ be the original units. Then (120)

$$e_1{}^2=e_2{}^2=\dots=e_n{}^2=1, \text{ and } 0=[e_1 \mid e_2]=\dots$$

131. PROJECTION.—DEFINITION.—If n is the order of the space considered, $a_1, \dots a_n$ are independent quantities of the first order, $A_1, A_2, \dots A_n$ are the

multiplicative combinations of these quantities of any one class, $A_1, \ldots A_m$ the multiplicative combinations of m of the same quantities $a_1, \ldots a_m$, and

$$A=\alpha_1 A_1+\alpha_2 A_2+\ldots \alpha_m A_m+\ldots \alpha_n A_n$$

$$A'=\alpha_1 A_1+\alpha_2 A_2+\ldots \alpha_m A_m.$$

A' is called the projection of A on the space $[a_1, a_2, \ldots a_m]$ by exclusion of the space $[a_{m+1}, \ldots a_n]$.

REMARK.—We have introduced here for want of a better the geometrical term *projection* to translate *Zurückleitung*. Literally *Zurückleitung* means "leading back."

132. The projection A' of a quantity A on a space B by exclusion of the space C is

$$A'=\frac{[B.AC]}{[BC]}.$$

PROOF.—Let the quantities be taken as in 131 and let

$$[a_1 \ldots a_m]=B,\ [a_{m+1} \ldots a_n]=C.$$

Then $[AC]=[(\alpha_1 A_1+\ldots+\alpha_m A_m+\alpha_{m+1}A_{m+1}+\ldots \alpha_n A_n)C]$.

But since $A_1, \ldots A_m$ are the combinations formed out of $a_1, \ldots a_m$ and $A_{m+1}, \ldots A_n$ those out of $a_1, \ldots a_n$ which are not at the same time combinations out of $a_1, \ldots a_m$, then must each of the quantities $A_{m+1}, \ldots A_n$ contain at least one of the factors of $a_{m+1}, \ldots a_n$, and thus must have a factor in common with C. Therefore the terms

$$\alpha_{m+1}A_{m+1}C, \ldots \alpha_n A_n C$$

are each equal to zero. (43) Hence

$$[AC]=[(\alpha_1 A_1+\ldots+\alpha_m A_m)C]=\alpha_1[A_1 C]+\ldots+\alpha_m[A_m C].$$

$$\therefore\ [B.AC]=\alpha_1[B.A_1 C]+\ldots+\alpha_m[B.A_m C].$$

Since now each of the quantities $A_1, \ldots A_m$ consists of factors which are contained in B, then is each of the same incident to B. Consequently, since the orders of B and C are together equal to n, by (72), we have

$$[B.A_1 C]=[BC]A_1, \ldots [B.A_n C]=[BC]A_n,$$

and therefore

$$[B.AC]=[BC](\alpha_1 A_1+\ldots+\alpha_m A_m)=[BC]A'.$$

Now since $[BC]$ is a number, we get

$$A'=\frac{[B.AC]}{[BC]}.$$

133. *If the projections taken in the same sense of the terms of an equation replace those terms, the result is a true equation.*

PROOF.—Let Q be the space on which the projection is made, R that excluded and $[QR]=1$. Then if the given equation is

$$P=\alpha A+\beta B+\ldots$$

$$[PR]=\alpha[AR]+\beta[BR]+\ldots$$

$$\text{and } [Q.PR]=\alpha[Q.AR]+\beta[Q.BR]+\ldots$$

$$\text{or, } P'=\alpha A'+\beta B'+\ldots$$

where P', A', . . . are the *projections* of terms in the given equation.

134. DEFINITION.—The projection A' of a quantity A on a space B by exclusion of the space $|B$ is called the *normal* projection.

From 132 we have for the normal projection

$$A'=\frac{[B.(A\,|\,B)]}{B^2}.$$

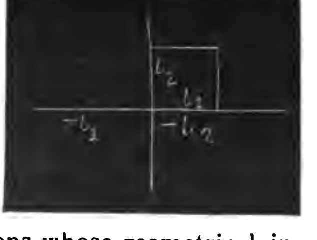
CHAPTER IX.

INNER PRODUCTS, NORMAL SYSTEMS, AND PROJECTION IN GEOMETRY.

135. Let ι_1 and ι_2 be two unit vectors constituting a simple normal system of the second order. Then by definition 125

$$\iota_1\iota_2=1, \text{ and } \iota_1\,|\,\iota_2=0,\ \iota_2\,|\,\iota_1=0.$$

We have also by definition of complement (57) $|\iota_1=\iota_2$ and $|\iota_2=-\iota_1$ since these values make $\iota_1\,|\,\iota_1=\iota_1\iota_2=1$, and $\iota_2\,|\,\iota_2=-\iota_2\iota_1=1$ (37). Also $||\iota_1=-\iota_1$ and $||\iota_2=-\iota_2$ (60).



Thus we see that taking the complement of a vector twice reverses it, *i. e.* revolves it through 180°, so that we are led to suppose that taking it once would revolve the vector through 90°. If this view of the complement can be shown to be consistent with the laws of the *Ausdehnungslehre*, we will adopt it.

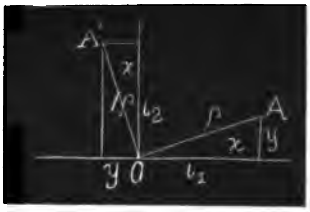
We have introduced above the following equations whose geometrical interpretation we append to each.

(1) $\iota_1\iota_2=1=$the unit of area (88).

(2) $|\iota_2 = -\iota_1$, *i. e.* taking the complement of ι_2 revolves it in the positive direction, opposite to the motion of the hands of a watch, into $-\iota_1$.

(3) $\iota_1 | \iota_2 = \iota_1(-\iota_1) = 0$ (34).

(4) Let $\rho = x\iota_1 + y\iota_2$ be any vector in the plane. Then, 58,



$$|\rho = x | \iota_1 + y | \iota_2 = x\iota_2 - y\iota_1.$$

The last value shows that $|\rho$ is OA', at right angles to OA. Thus here again taking the complement of a vector revolves it through 90° in the positive direction.

136. Comparing now the last part of the preceding article with 126-127 we see that the system whose units are ι_1 and ι_2 is transformed by circular alteration into that whose units are ρ and $|\rho$, provided $x^2 + y^2 = 1$, which makes the tensors of the new vectors each equal to unity. Thus circular alteration turns each of the units through the same angle in the same direction.

137. *If ε_1 and ε_2 are any two vectors, $\varepsilon_1 | \varepsilon_2 = 0$ is the condition that these two vectors are perpendicular to each other.*

For, $|\varepsilon_2$ denotes a vector perpendicular to ε_2 and $\varepsilon_1 | \varepsilon_2 = 0$ denotes that ε_1 and $|\varepsilon_2$ coincide.

138. Let ι_1, ι_2, ι_3 be three unit vectors constituting a simple normal system of the third order. Then by Definition 125

$$\iota_1\iota_2\iota_3 = 1, \quad \iota_1 | \iota_2 = 0, \quad \iota_1 | \iota_3 = 0, \quad \iota_2 | \iota_3 = 0.$$

We have also, by definition of complement (57),

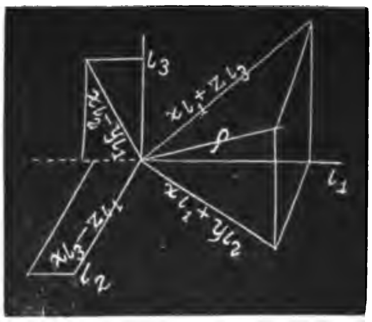
$$|\iota_1 = \iota_2\iota_3, \quad |\iota_2 = \iota_3\iota_1, \quad |\iota_3 = \iota_1\iota_2; \quad ||\iota_1 = \iota_1, \quad ||\iota_2 = \iota_2, \quad ||\iota_3 = \iota_3 \text{ (60)}.$$

Thus we see (89) that the complement of a line is a plane, and the complement of a plane is a line.

Let $\rho = x\iota_1 + y\iota_2 + z\iota_3$ = any line in space. Then, (58),

$$|\rho = x | \iota_1 + y | \iota_2 + z | \iota_3 = x[\iota_2\iota_3] + y[\iota_3\iota_1] + z[\iota_1\iota_2]$$

$$= \frac{1}{x}[(x\iota_2 - y\iota_1)(x\iota_3 - z\iota_1)] \text{ (38, 34).}$$



By 89 the right member equals the plane segment formed with $x\iota_2 - y\iota_1$ and $x\iota_3 - z\iota_1$.

Now ρ is perpendicular to each of these vectors and therefore perpendicular to their plane. For

$[(x\iota_1 + y\iota_2 + z\iota_3) | (x\iota_2 - y\iota_1)] = 0$, and

$$[(x\iota_1+y\iota_2+z\iota_3) \mid (x\iota_3-z\iota_1)]=0,$$

since, by hypothesis, $[\iota_1 \mid \iota_2]=0$, etc.

Hence the complement of a vector is a plane perpendicular to it.

139. PROJECTIONS.—*Let $\rho=x\varepsilon_1+y\varepsilon_2$ be given to find its projection on ε_1 and ε_2. respectively.*

Expressing ρ as the sum of its projections on ε_1 and ε_2, we have

$$\rho=\frac{[\varepsilon_1.\rho\varepsilon_2]}{[\varepsilon_1\varepsilon_2]}+\frac{[\varepsilon_2.\rho\varepsilon_1]}{[\varepsilon_2\varepsilon_1]}\ (132)=\frac{[\rho\varepsilon_2]}{[\varepsilon_1\varepsilon_2]}\varepsilon_1+\frac{[\rho\varepsilon_1]}{[\varepsilon_2\varepsilon_1]}\varepsilon_2$$

since $[\rho\varepsilon_2]$, and $[\rho\varepsilon_1]$ are scalars in plane space.

140. *To express ρ as the sum of its projections on any three vectors ε_1, ε_2, ε_3.*

$$\rho=\frac{[\rho\varepsilon_2\varepsilon_3]}{[\varepsilon_1\varepsilon_2\varepsilon_3]}\varepsilon_1+\frac{[\rho\varepsilon_3\varepsilon_1]}{[\varepsilon_1\varepsilon_2\varepsilon_3]}\varepsilon_2+\frac{[\rho\varepsilon_1\varepsilon_2]}{[\varepsilon_1\varepsilon_2\varepsilon_3]}\varepsilon_3 \text{ (By 132. See 8).}$$

141. *To express p as the sum of its projections on any four points p_1, p_2, p_3, p_4.*

$$p=\frac{[p\ p_2p_3p_4]}{[p_1p_2p_3p_4]}p_1-\frac{[p\ p_3p_4p_1]}{[p_1p_2p_3p_4]}p_2+\frac{[p\ p_4p_1p_2]}{[p_1p_2p_3p_4]}p_3-\frac{[p\ p_1p_2p_3]}{[p_1p_2p_3p_4]}p_4 \text{ (132).}$$

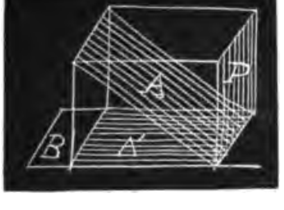
See Articles 95 and 9. The substitution of $p\rho$ for p (76) in the last equation may serve to throw light on this case of point projection.

142. Since the formula of 132 is general in its application, the quantities in the equation of 140 may be all points, or all vectors, or all lines, or all plane vectors. In the equation of 141 the points may all be replaced by planes.

143. Following Hermann Grassmann Jr. in his notes to the *Ausdehnungslehre* of 1862, we will illustrate the formula of 132 by some geometrical examples. We suppose the quantities considered situated in solid space (4th order).

(1) *To find the projection A' of A on B by exclusion of C where A and C are points and B is the plane segment, MN.*

A' is the point where CA pierces B taken such that $A=nC+A'$. For, multiplying both members of $A=nC+A'$ by C, we have $[AC]=[A'C]$. Again multiplying B by both members of the last equation, we get $[B.AC]=[B.A'C]$. But $[B.A'C]=[BC]A'$ (72); whence $A'=[B.AC]\div[BC]$. By symmetry nC is the projection of A on C by exclusion of B.



(2) *To find the projection A' of the plane segment A by exclusion of the point C.*

We have $A=A'+P$ where P is in the plane passing through C and the intersection of A and B.

Proof follows lines of (1). Begin by multiplying through by C.

We have also P is the projection of A on C by exclusion of B as in (1).

144. Suppose $[q_2q_3q_4]$ denotes a plane and $[p_1p_2]$ a line. Then their stereometric product is a scaler times their point of intersection (111). Now let

$$[p_1p_2\cdot q_2q_3q_4]=xp_1+yp_2=-[p_2q_2q_3q_4]p_1+[p_1q_2q_3q_4]p_2$$

by multiplying the members of the first equation by p_2 and p_1 in turn, thus getting values for x and y. Now multiply the members of the last equation by $|q_2$ and at the same time write $[q_2q_3q_4]$ as $|q_1$ (138). Then

$$[p_1p_2\,|q_1\,|q_2]=-[p_2\,|q_1][p_1\,|q_2]+[p_1\,|q_1][p_2\,|q_2]$$

$$\text{or } [p_1p_2\,|q_1q_2]=\begin{vmatrix} p_1\,|q_1 & p_1\,|q_2 \\ p_2\,|q_2 & p_2\,|q_2 \end{vmatrix} \text{ (55. 64).}$$

Putting $q_1=p_1$ and $q_2=p_2$ we have

$$[p_1p_2]^2=p_1{}^2p_2{}^2-[p_1\,|p_2]^2.$$

$p_1=1$ in the same equation, we have

$$[p_2\,|q_1q_2]=[p_2\,|q_2]\,|q_1-[p_2\,|q_1]\,|q_2.$$

This equation holds also when the p's are replaced by vectors.

145. Suppose $[p_1p_2p_3]$ denotes a plane and L a line. Then (111)

$$[p_1p_2p_3L]=xp_1+yp_2+zp_3=[p_2p_3L]p_1+[p_3p_1L]p_2+[p_1p_2L]p_3$$

by multiplying through by $[p_2p_3]$, $[p_3p_1]$, $[p_1p_2]$ in turn, thus getting values for x, y, z. Now for L put $|q_1q_2$ and multiply the members by $|q_3$. Then

$$[p_1p_2p_3\cdot|q_1q_2\cdot|q_3]=[p_2p_3\,|q_1q_2][p_1\,|q_3]$$

$$+[p_3p_1\,|q_1q_2][p_2\,|q_3]+[p_1p_2\,|q_1q_2][p_3\,|q_3],$$

$$\text{or. } [p_1p_2p_3\,|q_1q_2q_3]=\begin{vmatrix} p_1\,|q_1 & p_1\,|q_2 & p_1\,|q_3 \\ p_2\,|q_1 & p_2\,|q_2 & p_2\,|q_3 \\ p_3\,|q_1 & p_3\,|q_2 & p_3\,|q_3 \end{vmatrix} \text{ (55, 64, 144).}$$

In this equation planes may be substituted for points.

CHAPTER X.

146. Combinatory multiplication gives us a very neat solution of sets of simultaneous equations. Let n equations of the first degree containing n unknowns be given to find the value of the unknowns. Let the n equations be

$$\alpha_{11}x_1+\alpha_{21}x_2+\ldots\ldots+\alpha_{n1}x_n=\beta_1$$
$$\alpha_{12}x_1+\alpha_{22}x_2+\ldots\ldots+\alpha_{n2}x_n=\beta_2$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\alpha_{1n}x_1+\alpha_{2n}x_2+\ldots\ldots+\alpha_{nn}x_n=\beta_n$$

We multiply the first of these equations through by e_1, the second through by e_2, and so on, and the last by e_n, (where $[e_1e_2\ldots\ldots e_n]=1$), and add the resulting equations. Then if

$$\alpha_{11}e_1+\alpha_{12}e_2+\ldots\ldots+\alpha_{1n}e_n=a_1$$
$$\alpha_{21}e_1+\alpha_{22}e_2+\ldots\ldots+\alpha_{2n}c_n=a_2$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\alpha_{n1}e_1+\alpha_{n2}e_2+\ldots\ldots+\alpha_{nn}e_n=a_n$$
$$\beta_1e_1+\beta_2e_2+\ldots\ldots\ldots+\beta_ne_n=b,$$

we have for the sum of the products referred to above, the equation

$$x_1a_1+x_2a_2+\ldots\ldots+x_na_n=b.$$

By 20 this one equation replaces or transforms the set originally given us. Now in order to find x_1, multiply the last equation by $[a_2a_3\ldots\ldots a_n]$. This gives (52, 43)

$$x_1[a_1a_2\ldots\ldots a_n]=[ba_2a_3\ldots\ldots a_n]$$
$$\text{or, } x=\frac{[ba_2a_3\ldots\ldots a_n]}{[a_1\,a_2\ldots\ldots a_n]}$$

Replacing the a's and b by their values, by 45 we have the usual expression for x_1 in terms of the coefficients.

CHAPTER XI.

APPLICATION TO TRIGONOMETRY.

147. Definition.—The angle between two quantities is that angle ($<\pi$) whose cosine equals the inner product of the two quantities divided by the product of their scalar coefficients. Thus

$$\cos\angle ab=[a\mid b]\div\alpha\beta$$

where a and b are two quantities and α and β are their numerical values (123).

Again, if a, b, c.... are quantities of the first order, α, β, γ.... are their respective numerical values. $\sin(a\,b\,c\ldots.)$ is that numerical quantity which equals

$$\frac{[a\;b\;c\ldots.]}{\alpha\beta\gamma\;\ldots}$$

and is not negative. Thus (123) $\sin^2(abc)=\dfrac{[a\;b\;c\ldots.]^2}{\alpha^2\beta^2\gamma^2\ldots.}$.

148. *If a and b are quantities of the first order*, $\sin(ab)=\sin\angle ab$.

PROOF.—$\sin^2(ab)=\dfrac{[a\,b]^2}{\alpha^2\beta^2}=\dfrac{a^2b^2-[a\mid b]^2}{\alpha^2\beta^2}$ (144)

$$=\frac{\alpha^2\beta^2-[a\mid b]^2}{\alpha^2\beta^2}\ (123)=1-\frac{[a\mid b]^2}{\alpha^2\beta^2}\ (123)$$

$$=1-\cos^2\angle ab=\sin^2\angle ab\ (147).$$

Then if $\sin(ab)$ is never negative and $\angle ab<\pi$, $\sin(ab)=\sin\angle ab$.

149. *If α, β, γ, δ are the numerical values of a, b, c, d, by* 147 *and* 148,

$$[ab\mid cd]=\alpha\beta\gamma\delta\sin\angle ab\sin\angle cd\cos\angle(ab.cd).$$

150. *The normal projection of A on a quantity B of the same order is numerically equal to $A\cos\angle AB$.*

PROOF.—If A' is the normal projection of A on B (134)

$$A'=\frac{[A\mid B]B}{\beta^2}=\frac{\alpha\beta\cos\angle AB.B}{\beta^2}\ (119,\ 144)$$

$$=\alpha\cos\angle AB.\frac{B}{\beta}=\text{(numerically) } A\cos\angle AB.$$

151. The two expressions $[a\mid b]$ and $[ab]$, where a and b are vectors, play a very important part in mathematics. They occur yoked together in quaternions and apart in the *Ausdehnungslehre*, typifying the two products, the inner and outer. Numerically, as we have just seen, $[a\mid b]$ is the *projection* of either vector on the other multiplied by the tensor of the other; $[ab]$, on the other hand, is the area of the parallelogram whose adjacent sides are a and b, or, when the tensor of one vector is unity, it is equal numerically to the *perpendicular* from the extremity of the other vector on the first, when they go out from the same origin, or, when both tensors are unity, it is equal numerically to the sine of the angle between the given vectors (148).

152. *If a, b, c, ... are normal to each other and k is any quantity numerically derived from them, we have*

$$\frac{k}{\kappa}=\frac{a}{\alpha}\cos\angle ak+\frac{b}{\beta}\cos\angle bk+\ldots.$$

PROOF.—Let $k=xa+yb+\ldots.$ Then to find x multiply each member by $|a$. There results, since $[b\,|\,a]=0$, etc., $[k\,|\,a]=x[a\,|\,a]$. Finding the value of $y\ldots.$ in the same way and substituting we get the equation as given above.

153. If $a, b, c\ldots.$ are normal to one another and k and l are two quantities numerically derivable from $a, b, \ldots.$

$$\cos\angle kl=\cos\angle ak\cos\angle al+\cos\angle bk\cos\angle bl+\ldots.$$

PROOF.—From 147, we have

$$\cos\angle kl=\frac{[k\,|\,l]}{\kappa\lambda}=\left[\frac{k}{\kappa}\Big|\frac{l}{\lambda}\right]\equiv$$

$$[(\frac{a}{\alpha}\cos\angle ak+\frac{b}{\beta}\cos\angle bk+\ldots.)\,|\,(\frac{a}{\alpha}\cos\angle al+\frac{b}{\beta}\cos\angle bl+\ldots.)]\ (152)$$

$$=\frac{a^2}{\alpha^2}\cos\angle ak\cos\angle al+\frac{b^2}{\beta^2}\cos\angle bk\cos\angle bl+\ldots.$$

$\therefore\ \cos\angle kl=\cos\angle ak\cos\angle al+\cos\angle bk\cos\angle bl+\ldots.$

154. If $a, b, c, \ldots.$ are normal to each other and k is numerically derivable from them, we have by putting $l=k$ in 153

$$1=\cos^2\angle ak+\cos^2\angle bk+\ldots.$$

155. If $a, b, c, \ldots.$ are normal to each other, and k and l two quantities numerically derivable from them are normal to each other, 153 gives

$$0=\cos\angle ak\cos\angle al+\cos\angle bk\cos\angle bl+\ldots.$$

156. Writing in the formula of 144 a for p_1, b for p_2, c for q_1, d for q_2 gives

$$\sin\angle ab\sin\angle cd\cos\angle(ab.cd)=\cos\angle ac\cos\angle bd-\cos\angle bc\cos\angle ad.$$

In this formula if c and d are replaced respectively by a and c there results

$$\sin\angle ab\sin\angle ac\cos\angle(ab.ac)=\cos\angle bc-\cos\angle ba\cos\angle ac,$$

a familiar formula of spherical trigonometry.

157. The last formula of 145, by substituting a, b, c for p_1, p_2, p_3, and a, b, c for q_1, q_2, q_3 gives (147)

$$\sin^2(abc)=1-\cos^2\angle bc-\cos^2\angle ca-\cos^2\angle ab+2\cos\angle bc\cos\angle ca\cos\angle ab.$$

158. *The formula* $(a+b)^2=a^2+2[a\,|\,b]+b^2=\alpha^2+2\alpha\beta\cos\angle ab+\beta^2$ gives the familiar extension of the Pythagorean proposition.

159. Let a, b, c be plane segments whose sum is the fourth face of a tetraedron of which they are the other three (75). Then

$$(a+b+c)^2=a^2+b^2+c^2+2[b\,|\,c]+2[c\,|\,a]+2[a\,|\,b]$$
$$=\alpha^2+\beta^2+\gamma^2+2\beta\gamma\cos\angle bc+2\alpha\gamma\cos\angle ca+2\alpha\beta\cos\angle ab,$$

which is the extension of the preceding result to space.

In words:—The square of the base of any tetraedron is equal to the sum of the squares of the lateral faces dimished by twice the products of each pair of lateral faces times the cosine of the diedral angle between them.

CHAPTER XII.

APPLICATION TO ANALYTIC GEOMETRY.

160. Let p_1, p_2, p_3 represent three unit points, and suppose their product is unity (57). Then

$$[p_1p_2p_3]=[p_1(p_2p_3+p_3p_1+p_1p_2)]=1. \ (43)$$

But if p denote any other unit point in the plane of $[p_1p_2p_3]$, by 94 we may replace p_1 by p in this product, getting

$$[p(p_2p_3+p_3p_1+p_1p_2)]=[p\,|\,(p_1+p_2+p_3)]=1. \ (57,\ 58)$$

Let p_s denote a unit point which is the mean of the reference points. Then $p_1+p_2+p_3=3p_s$ (81). Substituting this value of $p_1+p_2+p_3$ in the equation above, we have

$$3[p\,|\,p_s]=1, \text{ or in solid space, } 4[p\,|\,p_s]=1.$$

161. *The equation* $p=xp_1+yp_2+zp_3$ *represents a straight line, provided* x, y, z *satisfy a linear equation, as* $ax+by+cz=0$.

To see this let us eliminate z. Then

$$p=\frac{1}{c}\{x(cp_1-ap_3)+y(cp_2-bp_3)\}.$$

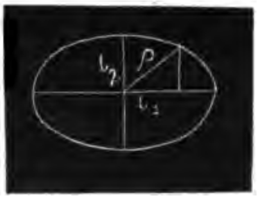
Thus, by 80, p lies on the right line through cp_1-ap_3 and cp_2-bp_3.

162. The equation $[p\,p_1p_2]=0$, in which p_1 and p_2 are constants and p is a variable is the equation of a straight line (94).

The equation $[pL]=0$, *where* p *is a point and* L *a line is the point equation*

of a straight line if p is variable and L is constant, and the line equation of the point p if L is variable and p is constant.

163. *The Cartesian equations of the central conics, the ellipse and the hyperbola in the inner product notation* (151) *are*



$$\left(\frac{\rho\,|\,\iota_1}{a}\right)^2 \pm \left(\frac{\rho\,|\,\iota_2}{b}\right)^2 = 1,$$

where ι_1 and ι_2 are unit vectors along the major and minor axes and ρ is the radius vector from the center to any point. Suppose we set

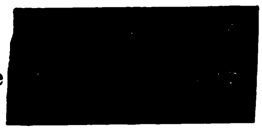
$$\frac{\rho\,|\,\iota_1}{a^2}\iota_1 \pm \frac{\rho\,|\,\iota_2}{b^2}\iota_2 \equiv \phi\rho.$$

Then the equation for the central conics reduces to

$$\rho\,|\,\phi\rho = 1.$$

164. DIFFERENTIATION.—Let ρ be a radius vector from an origin O to a curve AB. Then if ρ be made to approach indefinitely close to ρ_1, we have

$$\text{Limit}\frac{\rho-\rho_1}{AB} = \frac{d\rho}{ds} = a \textit{ unit vector}$$



in the direction of the tangent at A. This is taken to be the meaning of $\frac{d\rho}{ds}$ no matter whether ρ be a *vector* from the origin O, or a *point* moving from B to A on the curve AB.

165. The function ϕ (163) possesses the property that $\rho\,|\,\phi\rho_1 = \rho_1\,|\,\phi\rho$. Thus

$$\rho\,|\,\phi\rho_1 \equiv \rho\,|\left(\frac{\rho_1\,|\,\iota_1}{a^2}\iota_1 \pm \frac{\rho_1\,|\,\iota_2}{b^2}\iota_2\right) = \rho_1\,|\left(\frac{\rho\,|\,\iota_1}{a^2}\iota_1 \pm \frac{\rho\,|\,\iota_2}{b^2}\iota_2\right).$$

166. Differentiating the equation $\rho\,|\,\phi\rho=1$ (163), we get

$$d\rho\,|\,\phi\rho + \rho\,|\,\phi d\rho = 2d\rho\,|\,\phi\rho = 0. \text{ (165)}$$

Now if $d\rho$ is parallel to the tangent at the extremity of ρ, $\phi\rho$ is parallel to the normal (124).

If ρ_t and ρ_n be vectors to any point of the tangent and normal, respectively, and ρ_1 that to the point of contact, the equation of the tangent may be written $[(\rho_t-\rho_1)\,|\,\phi\rho_1]=0$, or $[\rho_t\,|\,\phi\rho_1]=1$, and that of the normal $[(\rho_n-\rho_1)\phi\rho_1]=0$. The ρ's may also be thought of as representing points.

167. Let p_1, p_2, p_3 denote the vertices of a reference triangle whose sides are of unit length and p any point in their plane. Then $|p_1=p_2p_3$, $|p_2=p_3p_1$,

$|\cdot p_3 = p_1 p_2$, and $p|p_1$, $p|p_2$, $p|p_3$ are proportional to the perpendiculars from p on the several sides of the triangle (94).

We shall consider only homogeneous equations. For, if any equation should not be homogeneous in p, all that is necessary to make it such is to introduce the factor $1 = 3p|p_s$ (160). Now the most general form of the equation of the second degree in trilinear coördinates is

$$a[p|p_1]^2 + b[p|p_2]^2 + c[p|p_3]^2 + 2d[p|p_2][p|p_3] + 2e[p|p_3][p|p_1] + 2f[p|p_1][p|p_2] = 0.$$

Let $$[(ap_1 + fp_2 + ep_3)p|p_1] + [(fp_1 + bp_2 + dp_3)p|p_2] + [(ep_1 + dp_2 + cp_3)p|p_3] \equiv \phi p.$$

When this value of ϕp is substituted in the preceding equation it reduces to $p|\phi p = 0$. Hence $p|\phi p = 0$ is the equation for all quadric curves whether central or non-central. Had quadriplaner coördinates been employed and the corresponding expressions constructed, an equation would have resulted representing any and all quadric surfaces. The same method may be used in getting the equation of the quadric in n-dimensional space.

REMARK.—The introduction of the ϕ function from Hamilton into the *Ausdehnungslehre* is due to Professor Hyde. (*Directional Calculus*, page 103). He shows that point analysis gives a means of changing the ordinary Cartesian equations into equations analogous to those of trilinear coördinates and then of generalizing the application of the equation $p|\phi p = 0$ to include the case of quadrics, central and non-central.

APPLICATION TO MECHANICS.

168. A force is completely represented by a point vector. We will denote a force by $F\rho$, where F denotes the length and direction of a vector, indicating respectively the intensity and direction of the force, and ρ is the point of application. It is apparent here that two letters are needed to properly represent the complex concept of a force. (See 76).

158. In Chapter I, we saw that the sum of two vectors is the diagonal of a parallelogram whose adjacent sides are the two given vectors. Then the sum of two forces, or their resultant, is the diagonal of a parallelogram whose two adjacent sides represent the two given forces. Similarly, all the results obtained for vectors in that chapter hold equally well for forces. The condition for equilibrium of forces acting on a particle at the extremity of ρ is evidently $(\Sigma F)\rho = 0$, or $\Sigma F = 0$.

169. The formulas obtained in Chapter VI evidently hold true when the points are replaced by infinitesimal forces, as parallel forces acting on particles, and also when they are replaced by finite parallel forces. (See 80).

170. Let the resultant of the forces acting on a rigid body be denoted by R. Then if $\epsilon = \rho - \rho_1$

$$R=\Sigma F\rho=\Sigma F\rho_1+\Sigma F(\rho-\rho_1)=(\Sigma F)\rho_1+(\Sigma F\varepsilon).$$

This result, called a "Wrench," contains two parts, a vector and a plane segment part. The vector ΣF represents the translation force, and the plane segment $\Sigma F\varepsilon$ gives the plane and magnitude of rotation. When the above result is interpreted geometrically, *i. e.* when ΣF is thought of as a line, and $\Sigma F\varepsilon$ as a plane segment, R is called a "Screw."

171. If R reduce to a single force, then by 34, $R^2=0$. We have then

$$R^2=(\Sigma F\rho_1+\Sigma F\varepsilon)^2=2(\Sigma F.\Sigma F\varepsilon)\rho_1=0.$$

This shows that ΣF and $\Sigma F\varepsilon$ must be parallel (112), *i. e.* the resultant force is parallel to the plane of the resultant couple. Evidently $\Sigma F.\Sigma F\varepsilon=0$ is satisfied also either by $\Sigma F=0$ whence $R=$a couple, or by $\Sigma F\varepsilon=0$, which makes of R a single force. If all of the forces lie in a single plane, the resultant can be reduced to either a single force or to a couple.

172. *For equilibrium we must have* (170), $\Sigma F=0$ *and* $\Sigma F\varepsilon=0$.

173. *A total resultant effect may be reduced to a single force and a couple whose place is perpendicular to the force by properly choosing the point of application.*

PROOF.—

$$R=\Sigma F\rho_1+\Sigma F(\rho-\rho_1)=\Sigma F\rho_2-\Sigma F(\rho_2-\rho_1)+\Sigma F(\rho-\rho_1).$$

Let $\Sigma F(\rho-\rho_1)=|\,\varepsilon$. Then

$$R=\Sigma F\rho_2-\Sigma F(\rho_2-\rho_1)+|\,\varepsilon.$$

Now the condition that $\Sigma F\rho_2$ shall be perpendicular to $|\,\varepsilon-\Sigma F(\rho_2-\rho_1)$ is

$$(|\,\varepsilon-\Sigma F(\rho_2-\rho_1)\,|\,\Sigma F\rho_2=|\,\varepsilon\,|\,\Sigma F\rho_2-\Sigma F(\rho_2-\rho_1).|\,\Sigma F\rho_2=0.$$

$\therefore\ |\,\Sigma F\rho_2.\Sigma(\rho_2-\rho_1)F=-\,|\,\Sigma F\rho_2\varepsilon$, (64, 38), whence

$$\frac{|\,\Sigma F\rho_2.\Sigma(\rho_2-\rho_1)F}{(\Sigma F\rho_2)^2}=-\frac{|\,\Sigma F\rho_2\varepsilon}{(\Sigma F\rho_2)^2}.$$

Comparing the left member of this equation with the formula of 132, we see that the right member is the value of the orthogonal projection of $\rho_2-\rho_1$ on a plane perpendicular to $\Sigma F\rho_2$.

Multiplying $\Sigma F\rho_2$ by the members of the last equation and interchanging the factors of $\Sigma(\rho_2-\rho_1)F$, we get

$$\Sigma F(\rho_2-\rho_1)=\frac{\Sigma F\rho_2 \,|\, \Sigma F\rho_2\varepsilon}{(\Sigma F\rho_2)^2}=\frac{|\,\Sigma F\rho_2 . \Sigma F\rho_2 \,|\, \varepsilon}{(\Sigma F\rho_2)^2}-|\,\varepsilon \text{ (144, last equation).}$$

Substituting the last value of $\Sigma F(\rho_2-\rho_1)$ in the first equation of this article, we have

$$R=\Sigma F\rho_2+\frac{\Sigma F\rho_2 \Sigma F(\rho_2-\rho_1)}{(\Sigma F\rho_2)^2}.|\,\Sigma F\rho_2,$$

which gives the required reduction.

CHAPTER XIV.

APPLICATION TO LOGIC.

174. The law of the inner product is $e_r e_s=0$. For in 117 if F is different from E, $[E\,|\,F]$ contains equal factors and is therefore zero, (43). This law, which is the opposite of that of the outer product (34), is made the basis of the study of spaces. Now a space in the *Ausdehnungslehre* corresponds to a *concept* or notion in logic. Hence the former science can be applied in the latter.

175. Definition.—The *Combined Space*, or the *sum* of two spaces, is the totality of quantities which belong to one or other of them (17).

176. Definition.—The *Common Space*, or the *product* of two spaces, is the totality of quantities which are common to both, (see 104). The product of two spaces which have no quantity of the first order common is zero (174).

Thus, using $^\circ(L_1L_2)$ and $^\circ(L_1L_3)$ to denote the two spaces which contain these lines, we see that L_1 is the common space.

177. *The sum of the orders m and n of two spaces equals the sum of the orders p and q of their common and combined spaces.*

Evidently $m+n$ duplicates the number expressing the order of the common space, and $p+q$ does the same.

178. *All the laws of space analysis continue to hold true in logic when the word "space" is replaced by the logical term "concept."*

Proof.—It is evident from the definitions 175, 176 that $^\circ(e+e)={}^\circ e$, and $^\circ(e.e)={}^\circ e$, and also that $^\circ e_r+{}^\circ e_s$ is greater or less than 0 and $^\circ(e_r).{}^\circ(e_s)=0$, when r is greater or less than s. But these are likewise the basic formulas of logic. (See H. and R. Grassmanns' *Formenlehre*, B. II., *Die Begriffslehre, oder Logik*,

1872, page 43. See also Encyclopedia Britannica article 'Logic,' section 35, paragragh 6.)

179. Two spaces can contain equal simple quantities only when they overlap. Thus if $A=abcd$ and $A'=a'b'c'd'$, these spaces have $a'bcd'$ in common, and quantities in $a'bcd'$ are in both A and A'. But E and F have no simple quantities in common. Since they are of the same size and lie in the same plane E and F in *geometry* would be equal (94); but in the theory of space or logic they are altogether different, having not one point, element, or bit of surface in common. It is highly important to note this difference if the reader is to avoid misconception.

180. *The associative and commutative laws for addition and multiplication hold for space analysis.*

PROOF.—It is evident from the definition 175 that $a+b=b+a$. Similarly from the definition in 176 it follows that $ab=ba$, and $abc=a(bc)$.

181. *Every sum, $n.{}^\circ A$, or product, $({}^\circ A)^n$, formed from the same space, ${}^\circ A$, equals this space.* This follows from proof in 178.

182. *One can add to any space a product of two spaces, one of whose factors is the given space, without altering its value.* Thus

$${}^\circ A={}^\circ A+{}^\circ A{}^\circ B.$$

One can multiply any space by a sum of two spaces, one of whose parts is the given space, without altering its value. Thus,

$${}^\circ A={}^\circ A({}^\circ A+{}^\circ B).$$

PROOF.—The product ${}^\circ A{}^\circ B$ is that part of ${}^\circ A$ which is common to both factors, 176. But adding a part of ${}^\circ A$ to ${}^\circ A$ gives ${}^\circ A$ by 181. In the other case we see that what is common to ${}^\circ A$ and ${}^\circ A+{}^\circ B$ is ${}^\circ A$.

183. *If ${}^\circ A+{}^\circ B={}^\circ B$, ${}^\circ A.{}^\circ B={}^\circ A$.*

PROOF.—${}^\circ A.{}^\circ B={}^\circ A({}^\circ A+{}^\circ B)$ (Hyp.)
$={}^\circ A$ (82)

If ${}^\circ A.{}^\circ B={}^\circ A$, ${}^\circ A+{}^\circ B={}^\circ B$.

PROOF.—${}^\circ A+{}^\circ B={}^\circ A{}^\circ B+B^\circ$ (Hyp.)
$={}^\circ A{}^\circ B+{}^\circ B^2$ (181)
$={}^\circ B({}^\circ A+{}^\circ B$ (Dist. Law)
$={}^\circ B$ (182)

184. *Unity added to any space gives unity, and any space multiplied by unity gives the same space; nought added to any space gives the same space, and any space multiplied by nought gives nought.*

PROOF.—The last three of these results are self-evident. From the first we have

$$1=1(1+1\times{}^\circ A) \quad (182)$$
$$=1+{}^\circ A \quad \text{(Dist. law and 2. of this Article.)}$$

185. *If ${}^\circ A+{}^\circ C={}^\circ B+{}^\circ C$ and also ${}^\circ A.{}^\circ C={}^\circ B.{}^\circ C$, ${}^\circ A={}^\circ B$.*

PROOF.—${}^\circ A={}^\circ A({}^\circ A+{}^\circ C)$ (182)$={}^\circ A({}^\circ B+{}^\circ C)$ (hyp.)$={}^\circ A.{}^\circ B+{}^\circ A{}^\circ C$ (Dist. law)

$$={}^\circ A{}^\circ B+{}^\circ B{}^\circ C(\text{hyp.})={}'B({}^\circ A+{}^\circ C)={}^\circ B({}^\circ B+{}^\circ C)(\text{hyp.})={}^\circ B \text{ (182).}$$

Definition.—The Non-Space or Complementary Space of ${}^\circ A$ is that space $|{}^\circ A$ (read non-A) which contains all the units not found in ${}^\circ A$, and none of the units which are found in ${}^\circ A$.

187. *The sum of a space ${}^\circ A$ and $|{}^\circ A$ is unity: the product of a space ${}^\circ A$ and $|A$ is zero.*

Proof.—The truth of the second part of the theorem follows directly from 176. For the other we write, by 184,

$${}^\circ A+|{}^\circ A=({}^\circ A+|{}^\circ A)|.$$

But by 176 the product of two spaces cantains all the quantities which are common to both. Then all the quantities of the left member ${}^\circ A+|{}^\circ A$ are contained in $|$. But ${}^\circ A$ and $|{}^\circ A$ contain all the units there are. Then $|$ contains all possible units.

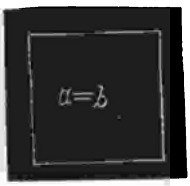
188. *All non-spaces of the same spaces are equal.*

Proof.—Suppose $|{}^\circ A$ and $|{}^\circ A_1$ to be two non-spaces of ${}^\circ A$. Then ${}^\circ A+|{}^\circ A=1={}^\circ A+|{}^\circ A_1$; whence by 185, remembering that ${}^\circ A.|{}^\circ A=0$ and ${}^\circ A.|{}^\circ A_1=0$, $|{}^\circ A=|{}^\circ A_1$.

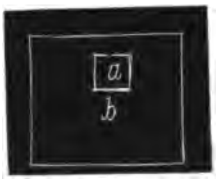
189. *The non-space of the non-space of a given space is the given space.*

Proof.—We have $|{}^\circ A+||{}^\circ A=1=|{}^\circ A+{}^\circ A$, and $||{}^\circ A.|{}^\circ A=0={}^\circ A\,|{}^\circ A$(187). Whence, by 185, $||{}^\circ A={}^\circ A$.

190. We give below four diagrams with descriptive names.

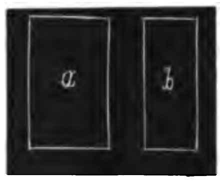


Identical Spaces.



Incident Space.

Cutting Spaces.



Disjunctive Spaces.

191. *Each of two spaces is incident to their sum. The product of two spaces is incident to each of them. Unity is the highest space, to which all other spaces are incident. Nought is the lowest space, being included in all other spaces.* Compare the use of unity here with its use in Chapter V.

The theorems already given will suffice to show that the language and subject matter of the *Ausdehnungslehre* can be utilized in the study of logic. The material for this chapter is taken from Robert Grassmann's *Ausdehnungslehre* (Slettin, 1891). Robert Grassmann aided his brother Hermann in the preparation of the *Ausdehnungslehre* of 1862 and the *Formenlehre* of 1872, and was always deeply interested in the subject.

The article in the Britannica already referred to and Professor Stokes' article in the January (1900) number of The American Mathematical Monthly show that there is considerable diversity of opinion regarding the philosophy underlying the application of mathematics to logic.